Math 112
Final Exam Solutions——

1. (20 pts.) Suppose the annual harvest of apples (in millions of boxes) in Washington is given by the function $A = f(T, R)$, where T is the average daily temperature in degrees Fahrenheit and R is the yearly rainfall in inches. The table below gives values of the function for particular values of T and R.

Average Daily Temperature T

	45	50	55
4	72.1	76.3	80.4
8	82.6	91.5	97
12	87.3	102.7	112.6

(a) (5 pts.) Find $f(50, 8)$ and interpret it in terms of the apple harvest.

Looking at the table ($T = 50$, $R = 8$), we have that $f(50, 8) = 91.5$. This means that when the average daily temperature is 50° Fahrenheit and the yearly rainfall is 8 inches, there is a harvest of 91.5 million boxes of apples.

(b) (8 pts.) Estimate $f_T(50, 8)$ and $f_R(50, 8)$ and interpret both in terms of the apple harvest.

- To estimate $f_T(50, 8)$: (Holding R constant.) The rate of change between $f(50, 8)$ and $f(45, 8)$ is $\frac{91.5-82.6}{50-45} = \frac{8.9}{5} = 1.78$. The rate of change between $f(50, 8)$ and $f(55, 8)$ is $\frac{97-91.5}{55-50} = \frac{5.5}{5} = 1.1$.

  Averaging the two values gives $f_T(50, 8) \approx 1.44$

  This means that if the temperature increases from 50° to 51° while the amount of rainfall stays constant at 8 inches, the harvest of apples will increase by approximately 1.44 million boxes.

- To estimate $f_R(50, 8)$: (Holding T constant.) The rate of change between $f(50, 8)$ and $f(50, 4)$ is $\frac{91.5-76.3}{8-4} = \frac{15.2}{4} = 3.8$. The rate of change between $f(50, 8)$ and $f(50, 12)$ is $\frac{102.7-91.5}{12-8} = \frac{11.2}{4} = 2.8$.

  Averaging the two values gives $f_R(50, 8) \approx 3.3$

  This means that if the yearly rainfall increases from 8 to 9 inches while the average daily temperature stays constant at 50°, the harvest of apples will increase by approximately 3.3 million boxes.

(c) (7 pts.) Estimate $f(53, 9)$ using the information from parts (a) and (b).

$$\begin{aligned} f(53,9) &\approx f(50,8) + 3f_T(50,8) + 1f_R(50,8) \text{ (Increasing } T \text{ by 3, Increasing } R \text{ by 1)} \\ &= 91.5 + 3(1.44) + 1(3.3) \\ &= 99.12 \text{ million boxes} \end{aligned}$$

2. (15 pts.) Find the average value of $g(t) = 4t^3(t^4 - 3)^4$ from $t = -1$ to $t = 2$.

The average value of $g(t)$ from $t = -1$ to $t = 2$ is given by $\frac{1}{2-(-1)} \int_{-1}^{2} 4t^3(t^4 - 3)^4 \, dt$

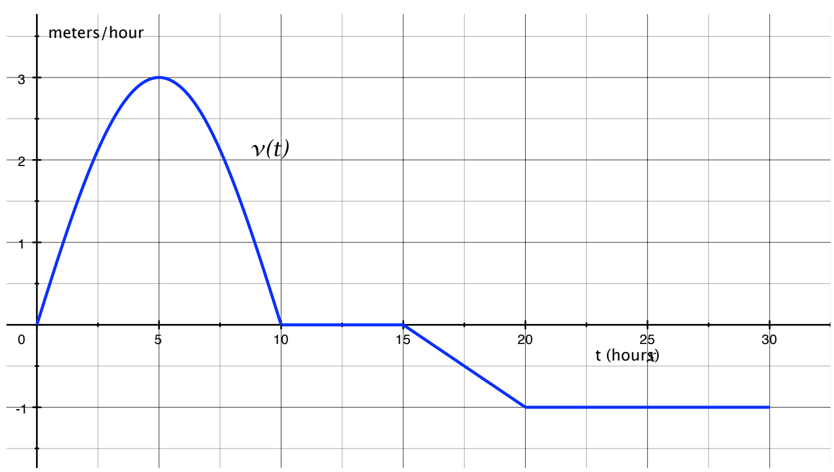
$$= \frac{1}{3} \int_{-1}^{2} 4t^3(t^4 - 3)^4 \, dt$$

Using substitution: $u = t^4 - 3 \quad \rightarrow \quad du = 4t^3 \; dt$, we have

$$\frac{1}{3} \int_{-1}^{2} 4t^3(t^4 - 3)^4 \, dt = \frac{1}{3} \int_{-2}^{13} u^4 \, du \quad (t = -1 \rightarrow u = (-1)^4 - 3 = -2, \quad t = 2 \rightarrow u = 2^4 - 3 = 13)$$

$$\begin{aligned} &= \frac{1}{3}\left(\frac{1}{5}u^5\right)\Big|_{-2}^{13} \\ &= \frac{1}{15}(13^5 - (-2)^5) \\ &= \frac{1}{12}(371325) = 24755 \end{aligned}$$

3. (25 pts.) The following graph of $v(t)$ gives the velocity of a sloth in meters/hour. Positive velocity indicates time at which the sloth travels towards a water hole.



(a) (5 pts.) Estimate time intervals for which the acceleration of sloth is positive.

Acceleration is the derivative of velocity. So, acceleration is positive when the slope of the velocity graph is positive, i.e. the velocity graph is increasing. $\Rightarrow$ Acceleration is positive from $t = 0$ to $t = 5$.

(b) (7 pts.) Write a definite integral that gives the total change of the sloth's position between 0 and 30 hours.

Since $v(t)$ = velocity of the sloth = rate of change of position of the sloth, $\int_0^{30} v(t)\,dt$ gives the total change of the sloth's position between $t = 0$ and $t = 30$.

(c) (8 pts.) Approximate the integral from part (a).

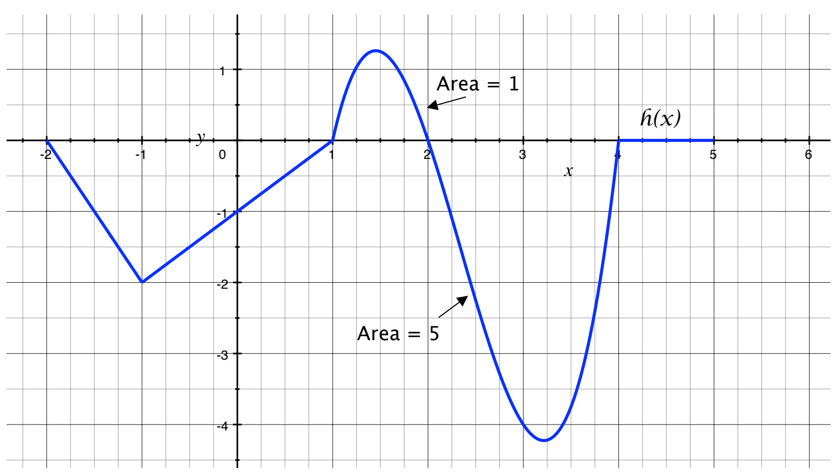
The area between the curve and the t-axis from $t = 0$ to $t = 10$ is approximately 18.75 and the area between the curve and the t-axis from $t = 15$ to $t = 30$ is 12.5.

So, $\int_0^{30} v(t)\,dt \approx 18.75 - 12.5 = 6.25$ meters.

(d) (5 pts.) If the sloth arrived at the water hole at $t = 10$, then what was the distance between the sloth and the water hole at time 0?

Between $t = 0$ and $t = 10$, the sloth traveled approximately 18.75 meters (area between curve and t-axis), so at time 0, the llama was approximately 18.75 meters away from the stream.

4. (15 pts.) Given the graph of $h(x)$ below, sketch a graph of $H(x)$ such that $H(1) = 1$ and $H'(x) = h(x)$ for $-2 \le x \le 5$. Include the function values of H at $x = -2,\ 1,\ 2,\ 4,$ and 5.



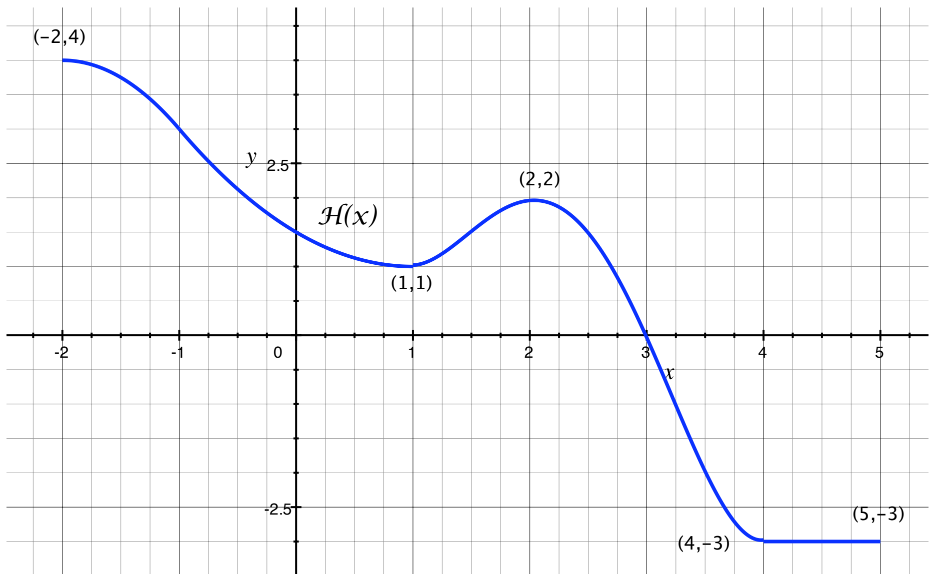
Since $H(1) = 1$, $H(2) = 1 + \int_1^2 h(x)\,dx = 1 + 1 = 2$

Since $H(2) = 2$, $H(4) = 2 + \int_2^4 h(x)\,dx = 2 - 5 = -3$

Since $H(4) = -3$, $H(5) = -3 + \int_4^5 h(x)\,dx = -3 + 0 = -3$

Since $H(1) = 1,\ H(1) = H(-2) + \int_{-2}^{1} h(x)\,dx = H(-2) - 3$

$\Rightarrow \quad H(-2) = H(1) + 3 = 1 + 3 = 4$



5. (15 pts.) For the function $z = f(x, y) = 4x^2 + 3x^3 \ln y + e^y$, find $\frac{\partial z}{\partial x}\Big|_{(2,3)}$ and $\frac{\partial z}{\partial y}\Big|_{(2,3)}$.

$$\frac{\partial z}{\partial x} = 8x + 9x^2 \ln y \text{ (Holding } y \text{ constant.)}$$

$$\frac{\partial z}{\partial y} = 3x^3 \frac{1}{y} + e^y \text{ (Holding } x \text{ constant.)}$$

So, $\frac{\partial z}{\partial x}\Big|_{(2,3)} = 8(2) + 9(2^2)\ln 3 = 16 + 36 \ln 3$ and $\frac{\partial z}{\partial y}\Big|_{(2,3)} = 3(2^3)\ \frac{1}{3} + e^3 = 8 + e^3$

6. (10 pts.) Find the consumer surplus for the demand curve $p = 250 - e^{.01q}$ when 300 units are sold.

If 300 units are sold ($q = 300$), then the price they are sold at is

$$p = 250 - e^{.01(300)} = 229.91 \text{ dollars.}$$

The amount consumers are willing to pay for the 300 units is $\int_0^{300} 250 - e^{.01q}\,dq$

$$= 250q - \frac{e^{.01q}}{.01}\Big|_0^{300}$$

$$= 73091.45 \text{ dollars}$$

The amount that consumers actually paid is $(229.91)(300) = 68973$ dollars.

So, the consumer surplus is 73091.45 - 68973 = 4118.45 dollars.