# Math 120
# Exam 2 Solutions

1. (14 pts.) Find the exact values of the following expressions.

   (a) (6 pts.) $\cos^{-1}(-\frac{1}{2})$

   Note that $\cos^{-1}(-\frac{1}{2})$ is an angle $\theta$ between 0 and $\pi$ such that $\cos\theta = -\frac{1}{2}$.

   Since $\cos(\frac{2\pi}{3}) = -\frac{1}{2}$, $\cos^{-1}(-\frac{1}{2}) = \frac{2\pi}{3}$.

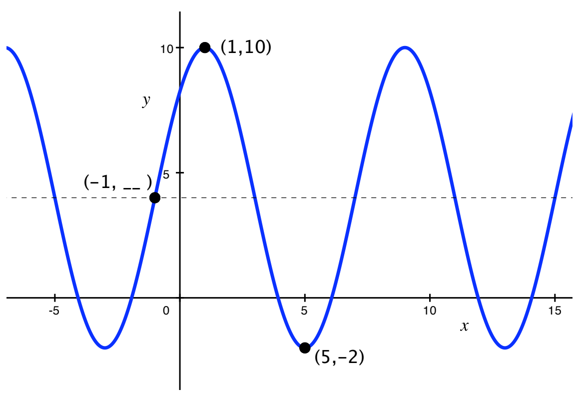
   (b) (8 pts.) $\sin^{-1}(\sin(\frac{10\pi}{3}))$

   Note that $\sin(\frac{10\pi}{3}) = -\frac{\sqrt{3}}{2}$.

$$\Rightarrow \quad \sin^{-1}(\sin(\frac{10\pi}{3})) = \sin^{-1}(-\frac{\sqrt{3}}{2}) = -\frac{\pi}{3}$$

   since $\theta = -\frac{\pi}{3}$ is an angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ such that $\sin\theta = -\frac{\sqrt{3}}{2}$.

2. (19 pts.) The graph of a sinusoidal function $f(x)$ is given below.




(a) (9 pts.) Find the amplitude, mean, and period.

The maximum value of $f(x)$ is 10 and the minimum is $-2$.

Amplitude $= \frac{1}{2}[10 - (-2)] = 6$

Mean $= \frac{1}{2}[10 + (-2)] = 4$

Note that the maximum occurs at $x = 1$ and the next minimum occurs at $x = 5$. Since the difference of $x$-values between a maximum and minimum is 4 (a half-period), the period is 8.

(b) (10 pts.) Write an equation for $f(x)$.

Note that since the period is 8 units, we have that $k = \dfrac{2\pi}{8} = \dfrac{\pi}{4}$.

If using the form $y = a\sin k(x-b) + c$: $\quad y = 6\sin\frac{\pi}{4}(x+1) + 4$
since we can use $b = -1$ as the phase shift for sine.

If using the form $y = a\cos k(x-b) + c$: $\quad y = 6\cos\frac{\pi}{4}(x-1) + 4$
since we can use $b = 1$ as the phase shift for cosine.

3. (15 pts.) Prove the following identity: $\cot^2 x - \cos^2 x = \cot^2 x \cdot \cos^2 x$

Here are a couple ways to prove this identity:

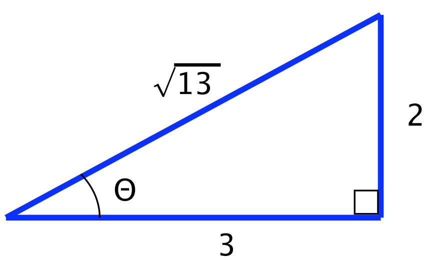
- $$\begin{aligned} \text{RHS} = \cot^2 x \cdot \cos^2 x &= \cot^2 x(1 - \sin^2 x) \\ &= \cot^2 x - \cot^2 x \cdot \sin^2 x \\ &= \cot^2 x - \frac{\cos^2 x}{\sin^2 x} \cdot \sin^2 x \\ &= \cot^2 x - \cos^2 x = \text{LHS} \end{aligned}$$

  Similarly, you can use $\cot^2 x = \csc^2 x - 1$ on the RHS.

- $$\begin{aligned} \text{LHS} = \cot^2 x - \cos^2 x &= \frac{\cos^2 x}{\sin^2 x} - \cos^2 x \\ &= \frac{\cos^2 x - \cos^2 x \cdot \sin^2 x}{\sin^2 x} \\ &= \frac{\cos^2 x(1 - \sin^2 x)}{\sin^2 x} \\ &= \frac{\cos^2 x \cdot \cos^2 x}{\sin^2 x} \\ &= \frac{\cos^2 x}{\sin^2 x} \cdot \cos^2 x \\ &= \cot^2 x \cdot \cos^2 x = \text{RHS} \end{aligned}$$

4. (16 pts.) Find the exact value of $\cos(2\theta)$ given that $\tan \theta = \frac{2}{3}$ for $\theta$ in quadrant III.

If $\tan\theta = \frac{2}{3}$, we have the following reference triangle.




Since $\theta$ is in quadrant III, we have that $\cos\theta = -\frac{3}{\sqrt{13}}$ and $\sin\theta = -\frac{2}{\sqrt{13}}$.

We can use any form of the double-angle formula for cosine to find $\cos(2\theta)$.

Using the form $\cos(2\theta) = \cos^2\theta - \sin^2\theta$, we have that
$$\begin{aligned} \cos(2\theta) &= \left(-\frac{3}{\sqrt{13}}\right)^2 - \left(-\frac{2}{\sqrt{13}}\right)^2 \\ &= \frac{9}{13} - \frac{4}{13} \\ &= \frac{5}{13} \end{aligned}$$

5. (18 pts.) Find the exact values of the following using **angle addition/subtraction** formulas or **half-angle** formulas.

(a) (9 pts.) $\cos 67.5^{\text{o}}$

Note that if $\theta = 135^{\text{o}}$, then $\frac{\theta}{2} = 67.5^{\text{o}}$.

Using the half-angle formula, we have that

$$\begin{aligned}\cos 67.5^{\text{o}} &= \pm\sqrt{\tfrac{1+\cos 135^{\text{o}}}{2}} \\ &= \pm\sqrt{\tfrac{1+(-\frac{\sqrt{2}}{2})}{2}} \\ &= \pm\frac{1}{2}\sqrt{2-\sqrt{2}}\end{aligned}$$

Since $67.5^{\text{o}}$ is in quadrant I, $\cos 67.5^{\text{o}}$ is positive, so $\cos 67.5^{\text{o}} = \frac{1}{2}\sqrt{2-\sqrt{2}}$.

(b) (9 pts.) $\cos(\frac{5\pi}{12})$

Note that

$$\begin{aligned}\cos(\tfrac{5\pi}{12}) = \cos(\tfrac{\pi}{6}+\tfrac{\pi}{4}) &= \cos(\tfrac{\pi}{6})\cdot\cos(\tfrac{\pi}{4}) - \sin(\tfrac{\pi}{6})\cdot\sin(\tfrac{\pi}{4}) \\ &= \tfrac{\sqrt{3}}{2}\cdot\tfrac{\sqrt{2}}{2} - \tfrac{1}{2}\cdot\tfrac{\sqrt{2}}{2} \\ &= \tfrac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

The half-angle formula for cosine can also be used as well since if $\theta = \frac{5\pi}{6}$, then $\frac{\theta}{2} = \frac{5\pi}{12}$.

6. (18 pts.) Suppose the number of visible sunspots is given by the function

$$s(t) = 50\cos\tfrac{2\pi}{11}(t-3) \ + \ 60$$

where $t$ is the number of years after 2000. ($t = 0$ is the year 2000, $t = 1$ is the year 2001, etc.)

(a) (6 pts.) Find the period of $s(t)$.

The period is given by $P = \frac{2\pi}{\frac{2\pi}{11}} = 11$ years.

(b) (6 pts.) What are the maximum and minimum number of visible sunspots?

Since the mean is 60 and the amplitude is 50, the maximum number of visible sunspots is 60 + 50 = 110 and the minimum number is 60 − 50 = 10.

(c) (6 pts.) Find the first year after 2000 for which the number of visible sunspots is maximum. (Hint: Consider the graph of $s(t)$.)

Note that since the phase shift of this cosine function is 3, we have that the function has a maximum at $t = 3$, which corresponds to 2003. Since the period is 11 years, this is the first year after 2000 in which the number of visible sunspots is maximum.