Math 120
Final Exam Solutions

1. (a) Note that $\theta = -\frac{29\pi}{6} = -\frac{24\pi}{6} - \frac{5\pi}{6} = -4\pi - \frac{5\pi}{6}$
   So, θ is in quadrant III and the reference angle is $\overline{\theta} = \frac{\pi}{6}$.

   (b) $x = -3\cos(-\frac{29\pi}{6}) = -3(-\frac{\sqrt{3}}{2}) = \frac{3\sqrt{3}}{2}$

   $y = -3\sin(-\frac{29\pi}{6}) = -3(-\frac{1}{2}) = \frac{3}{2}$ Point: $(\frac{3\sqrt{3}}{2}, \frac{3}{2})$

2. (a) $2\mathbf{v} = <-4, 8>$ $\Rightarrow$ $\mathbf{u} - 2\mathbf{v} = <5, -3>$

   $\Rightarrow$ $(\mathbf{u} - 2\mathbf{v}) \cdot \mathbf{v} = -2 \cdot 5 + 4 \cdot (-3) = -22$

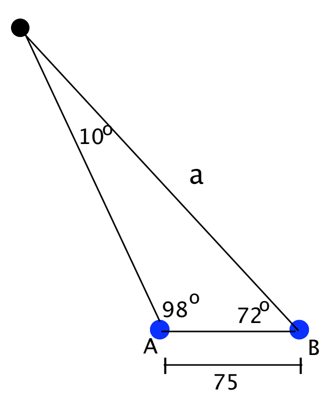
   (b) (7 pts.) Find the angle between **u** and **v**.

   We can use the dot product of **u** and **v** to find the angle θ between the two vectors.
   Note: $\mathbf{u} \cdot \mathbf{v} = 1 \cdot (-2) + 5 \cdot 4 = 18$ $|\mathbf{u}| = \sqrt{1^2 + 5^2} = \sqrt{26}$
   $|\mathbf{v}| = \sqrt{(-2)^2 + 4^2} = \sqrt{20}$

   $\Rightarrow$ $18 = \sqrt{26}\sqrt{20}\cos\theta$ or $\cos\theta = \frac{18}{\sqrt{26}\sqrt{20}}$

   $$\Rightarrow \theta = \cos^{-1}\left(\frac{18}{\sqrt{26}\sqrt{20}}\right) \approx 0.66104 \approx 37.8750^{o}$$

3. (a)



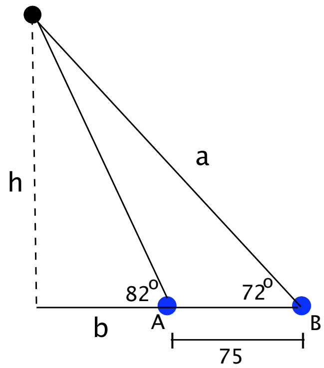
Consider the triangle formed with the satellite and stations as vertices.

The angle at station A is $180^o - 82^o = 98^o$, so the angle at the satellite is
$$180^o - 98^o - 72^o = 10^o$$

Let a = distance between the satellite and station B.

Law of Sines: $\frac{\sin 10^o}{75} = \frac{\sin 98^o}{a}$ $\Rightarrow$ $a = \frac{75\sin 98^o}{\sin 10^o} \approx 427.7045$ miles

(b) We are trying to find h in the following diagram:



Two ways to approach this problem:

- Given part (a), we have that $a \approx 427.7045$ So, $\sin 72^{o} \approx \dfrac{h}{427.7045}$

$$\Rightarrow \quad h \approx 427.7045 \sin 72^{o}$$
$$h \approx 406.7711 \text{ miles}$$

(Exact Value: $h = \dfrac{75 \sin 98^{o} \sin 72^{o}}{\sin 10^{o}}$)

- If we are not using the information from part (a), we can find h as follows.
  Note that $\tan 82^{o} = \dfrac{h}{b}$ and $\tan 72^{o} = \dfrac{h}{b+75}$.

$$\Rightarrow \quad b \tan 82^{o} = (b+75) \tan 72^{o} \qquad \text{(Solve for } h \text{ in each equation.)}$$

$$\Rightarrow \quad b = \frac{75 \tan 72^{o}}{\tan 82^{o} - \tan 72^{o}} \approx 57.1680 \text{ miles}$$

  So, $h = b \tan 82^{o} \approx 406.7711$ miles.

4. (a) $\sin^{-1}(-\frac{\sqrt{3}}{2}) = -\frac{\pi}{3}$ since $-\frac{\pi}{3}$ is the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ for which sine is equal to $-\frac{\sqrt{3}}{2}$.

   (b) $\cos^{-1}(\cos(\frac{35\pi}{6})) = \frac{\pi}{6}$ since $\cos(\frac{35\pi}{6}) = \frac{\sqrt{3}}{2}$, and $\frac{\pi}{6}$ is the angle between 0 and π for which cosine is equal to $\frac{\sqrt{3}}{2}$.

5. (a) The amplitude of the sinusoidal function is 50, so the maximum height is $58 + 50 = 108$ feet and the minimum is $58 - 50 = 8$ feet.

   The radius of the wheel is the same as the amplitude of the function, which is 50 feet.

   (b) The amount of time it takes for the wheel to make one complete revolution is the same as the period of the function. Period $= \dfrac{2\pi}{\frac{\pi}{60}} = 120$ seconds

(c) It takes 120 seconds for the wheel to make one complete revolution, so

$$\omega = \frac{2\pi}{120} = \frac{\pi}{60} \text{ radians/second.}$$

The linear speed is $\nu = r\omega = 50(\frac{\pi}{60}) = \frac{5\pi}{6}$ feet/second.

(d) If the height is 33 feet, then we are trying to solve for t (between 0 and 240 seconds) when

$$33 = 50\sin\tfrac{\pi}{60}(t-30) + 58 \qquad \Rightarrow \qquad \sin\tfrac{\pi}{60}(t-30) = -\tfrac{1}{2}$$

There are two possibilities:

- $$\begin{aligned} \tfrac{\pi}{60}(t-30) &= -\tfrac{\pi}{6} + 2n\pi && \text{for an integer } n \\ t-30 &= -10+120n && \text{(Dividing both sides by } \tfrac{\pi}{60}) \\ t &= 20+120n \end{aligned}$$

  Note: If $n=0$, $\Rightarrow$ $t=20$. If $n=1$, $\Rightarrow$ $t=140$.
  If $n \geq 2$ or if $n < 0$, t is not between 0 and 240 seconds.

- $$\begin{aligned} \tfrac{\pi}{60}(t-30) &= \tfrac{5\pi}{6} + 2n\pi \\ t-30 &= 70+120n \\ t &= 100+120n \end{aligned}$$

  Note: If $n=0$, $\Rightarrow$ $t=100$. If $n=1$, $\Rightarrow$ $t=220$.
  If $n \geq 2$ or if $n < 0$, t is not between 0 and 240 seconds.

So, we have the four times $t=20$, $t=100$, $t=140$, and $t=220$ seconds.

6. Here is one way to prove this identity.
Working with the left-hand side and getting a common denominator of $\sin\theta(1-\cos\theta)$:

$$\begin{aligned} \frac{1-\cos\theta}{\sin\theta} + \frac{\sin\theta}{1-\cos\theta} &= \frac{(1-\cos\theta)^2}{\sin\theta(1-\cos\theta)} + \frac{\sin^2\theta}{\sin\theta(1-\cos\theta)} \\ &= \frac{1-2\cos\theta+\cos^2\theta+\sin^2\theta}{\sin\theta(1-\cos\theta)} \\ &= \frac{1-2\cos\theta+1}{\sin\theta(1-\cos\theta)} \qquad \text{(Pythagorean Identity)} \\ &= \frac{2-2\cos\theta}{\sin\theta(1-\cos\theta)} \\ &= \frac{2(1-\cos\theta)}{\sin\theta(1-\cos\theta)} \\ &= \frac{2}{\sin\theta} \quad = \quad \text{Right-hand side} \end{aligned}$$

7. Using the double-angle formula on the left-hand side of the equation, we have

$$2\sin x\cos x = \tfrac{1}{3}\cos x$$

Here are two ways to solve this equation:

- Note that if $\cos x = 0$, then the equation would be satisfied. So, $x = \frac{\pi}{2} + n\pi$ is a set of solutions.

  If $\cos x \neq 0$, then we can divide both sides of the equation by $\cos x \quad \Rightarrow \quad 2\sin x = \frac{1}{3}$

  $$\Rightarrow \quad \sin x = \frac{1}{6} \quad \Rightarrow \quad x = \sin^{-1}(\tfrac{1}{6}) + 2n\pi \approx 0.1674 + 2n\pi$$

  $$\text{OR} \quad x = \pi - \sin^{-1}(\tfrac{1}{6}) + 2n\pi \approx 2.9741 + 2n\pi$$

- $$2\sin x\cos x - \tfrac{1}{3}\cos x = 0 \qquad \text{(Placing all terms on one side to factor.)}$$
  $$\cos x(2\sin x - \tfrac{1}{3}) = 0$$

  Zero-Factor Property:
  1) $\cos x = 0 \quad \Rightarrow \quad x = \frac{\pi}{2} + n\pi$

  2) $\sin x = \frac{1}{6} \quad \Rightarrow \quad x = \sin^{-1}(\tfrac{1}{6}) + 2n\pi \approx 0.1674 + 2n\pi$

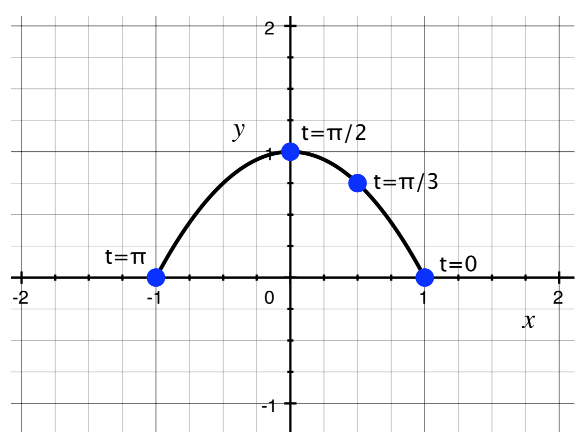
  $$\text{OR} \quad x = \pi - \sin^{-1}(\tfrac{1}{6}) + 2n\pi \approx 2.9741 + 2n\pi$$

So, there are 3 sets of solutions: $x = \frac{\pi}{2} + n\pi, \quad x = \sin^{-1}(\frac{1}{6}) + 2n\pi, \quad x = \pi - \sin^{-1}(\frac{1}{6}) + 2n\pi$

8. (a) Given that $0 \le t \le \pi$, cosine and sine can range in values from -1 to 1.
   So, the range of x-values is $-1 \le x \le 1$ and the range of y-values is $0 \le y \le 1$.
   (Range of $\sin^2 t$ is positive)

(b)

t	x	y
0	1	0
$\pi/3$	$\frac{1}{2}$	$\frac{3}{4}$
$\pi/2$	0	1
π	-1	0



(c) Here are two ways to eliminate the parameter:

- Note that $\cos^2 t + \sin^2 t = 1 \qquad \Rightarrow \qquad x^2 + y = 1 \qquad \text{or} \qquad y = 1 - x^2$

- Note that since $0 \le t \le \pi$, we have that $t = \cos^{-1} x$.

  $$\Rightarrow \qquad y = \sin^2(\cos^{-1} x)$$

  Using a reference triangle, we can simplify $\sin(\cos^{-1} x)$ into $\sqrt{1 - x^2}$.

  $$\Rightarrow \qquad y = \sin^2(\cos^{-1} x) = (\sqrt{1 - x^2})^2 = 1 - x^2 \qquad \text{for } -1 \le x \le 1$$