Math 111
Exam 3 Solutions

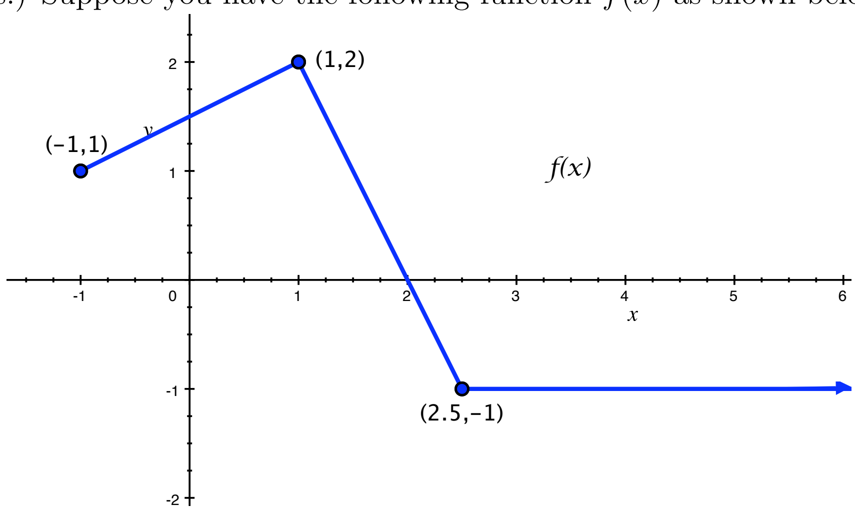
1. (14 pts.) Find an equation for the function $f(x)$ whose graph is a parabola that goes through the origin with vertex $(-2, 3)$.

   Using the vertex form of a quadratic equation: $f(x) = a(x - (-2))^2 + 3 = a(x+2)^2 + 3$

   Using the point $(0, 0)$ to find a: $0 = f(0) = a(0+2)^2 + 3 = 4a + 3 \quad \Rightarrow \quad -3 = 4a$
   $\Rightarrow \quad a = -\frac{3}{4}$

   So, $f(x) = -\frac{3}{4}(x+2)^2 + 3$.

2. (28 pts.) Suppose you have the following function $f(x)$ as shown below.



   (a) (8 pts.) What is the domain of $f(-\frac{1}{2}x)$?

   The domain of $f(x)$ is $x \geq -1$ or $[-1, \infty)$.

   The function $f(-\frac{1}{2}x)$ is $f(x)$ after the following transformations have been applied:
   1) Reflection over the y-axis
   2) Horizontal stretch by a factor of 2

   If we consider the effect of the reflection alone, we have that the new domain would be $x \leq 1$ or $(-\infty, 1]$.

   After the horizontal stretch, the domain would be $x \leq 2$ or $(-\infty, 2]$.

   So, the domain of $f(-\frac{1}{2}x)$ is $x \leq 2$ or $(-\infty, 2]$.

(b) (8 pts.) What is the range of $f(x) + 2$?

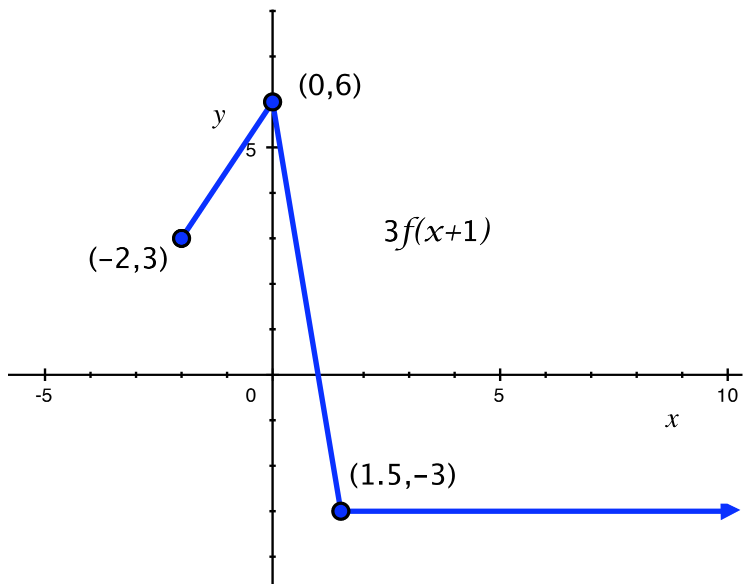
The range of $f(x)$ is $-1 \leq f(x) \leq 2$.

The function $f(x) + 2$ is the $f(x)$ shifted up by 2 units.

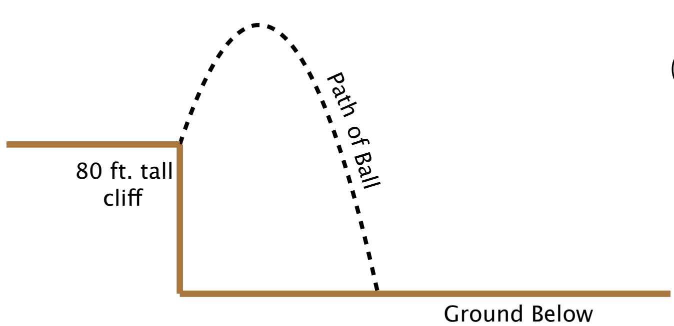
So, the range of $f(x) + 2$ will be $1 \leq f(x) + 2 \leq 4$ or $[1, 4]$.

(c) (12 pts.) Sketch the graph of $3f(x + 1)$ below. Clearly label the points on your graph.

The function $3f(x+1)$ is the function $f(x)$ after the following transformations have been applied:
1) Shift left by 1 unit
2) Vertical stretch by a factor of 3



3. (22 pts.) Ernie kicks a soccer ball up into the air with an initial velocity of 64 feet/second from the edge of a 80 foot cliff. The height of the ball above the ground below is given by $h = -16t^2 + 64t + 80$ in feet at t seconds.



(a) (10 pts.) When does the ball land?

When the ball lands, the height above the ground below will be 0 feet.

So, we need to solve
$-16t^2 + 64t + 80 = 0.$

Dividing both sides by -16: $t^2 - 4t - 5 = 0$

$$(t-5)(t+1)=0 \quad \Rightarrow \quad t=5 \quad \text{or} \quad t=-1$$

The only possible answer is that the ball lands at 5 seconds.

(b) (12 pts.) When does the ball reach it's maximum height? What is the maximum height? The maximum height of the ball will be given by finding the vertex of the parabola describing the height of the ball.

Using the vertex formula, the time at which the maximum height occurs is given by

$$t=-\frac{b}{2a}=-\frac{64}{2(-16)}=2 \text{ seconds.}$$

The maximum height is given by $h(2)=144$ feet.

4. (10 pts.) Decompose the function $g(x)=\frac{1}{\sqrt{2^x+5}}$ into two functions $u(x)$ and $v(x)$ such that $u(v(x))=g(x)$ (with $u(x)\neq x$ and $v(x)\neq x$).

Here are some possible decompositions.

- $u(x)=\frac{1}{x}, \quad v(x)=\sqrt{2^x+5}$
- $u(x)=\frac{1}{\sqrt{x}}, \quad v(x)=2^x+5$
- $u(x)=\frac{1}{\sqrt{x+5}}, \quad v(x)=2^x$

5. (13 pts.) You decide to buy a \$22,000 hybrid vehicle today. You make a down payment of \$5000 and get a 5-year loan for the rest of the balance with a 8.4% annual rate compounded monthly. How much do you have to pay monthly for the loan?

Since you are making a \$5000 down payment, you will only need to take out a loan of \$17000. Since the \$17000 is the value of the loan today, we will use the formula for the present value of an annuity.

Quantities: $P=17000$ $\quad R=?$ $\quad r=.084$ $\quad k=12$ $\quad t=5$

$$\Rightarrow \quad 17000=R\Big[\frac{1-(1+\frac{.084}{12})^{-12(5)}}{\frac{.084}{12}}\Big]=R\Big[\frac{1-1.007^{-60}}{.007}\Big]$$

$$\Rightarrow \quad R=347.96 \text{ dollars}$$

So, the monthly payment will be \$347.96.

6. (13 pts.) Suppose you want to retire in 40 years with 1 million dollars. You have a retirement account with an 8.4% annual rate compounded quarterly. How much would you have to invest every 3 months to reach your goal?

We will be using the formula for the future value of an annuity for this problem since 1 million dollars is the amount that we want in the future.

Quantities: $S = 1,000,000$ $\quad R = ?$ $\quad r = .084$ $\quad k = 4$ $\quad t = 40$

$$\Rightarrow \quad 1,000,000 = R\left[\frac{(1+\frac{.084}{4})^{4(40)}-1}{\frac{.084}{4}}\right] = R\left[\frac{1.021^{160}-1}{.021}\right]$$

$$\Rightarrow \quad R = 783.44 \text{ dollars}$$

So, the quarterly investment will be \$783.44.