Picard's Theorem Consider the first order initial value problem $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$. If $f(x,y)$ and $f_y(x,y)$ are continuous in some open rectangle $\{(x,y) \in R^2 \mid a < x < b, c < y < d\}$ containing (x_0, y_0), then the initial value problem has a unique solution in some open interval $x_0 - h < x < x_0 + h$ contained in $a < x < b$.

1. Consider the initial value problem $\frac{dy}{dx} = 3x^{2/3}$, $y(0) = 0$.
   i. Do the hypothesis of the above theorem apply? What can we conclude?
   ii. Solve the differential equation. What is the interval of existence of the solution?

2. Consider the initial value problem $\frac{dy}{dx} = 3y^{2/3}$, $y(0) = 0$.
   i. Do the hypothesis of the above theorem apply? What can we conclude?
   ii. Find solutions to the IVP.

3. Consider the initial value problem $\frac{dy}{dx} = 3y^{2/3}$, $y(1) = 8$.
   i. Do the hypothesis of the above theorem apply? What can we conclude?
   ii. Verify that $y = (x+1)^3$ is a solution to the IVP. What is the interval of existence of the solution?
   iii. Find other solutions to the IVP.

4. Consider the initial value problem $\frac{dy}{dx} = 2xy$, $y(0) = a$.
   i. Do the hypothesis of the above theorem apply? What can we conclude?
   ii. Verify that the function $y = ae^{x^2}$ is a solution to the IVP. What is the interval of existence of the solution? Over what interval is this solution unique?

5. Consider the initial value problem $\frac{dy}{dx} = 2xy^2$, $y(0) = a^2$, $a \geq 0$.
   i. Do the hypothesis of the above theorem apply? What can we conclude?
   ii. Verify that $y = \frac{a^2}{a^2x^2-1}$ is a solution. What is the interval of existence of the solution? What happens to the domain of the solution as $a \to \infty$?

6a. Sketch the direction field for the differential equation $\frac{dy}{dt} = -2(y-20)$
b. Sketch a solution to the IVP $\frac{dy}{dt} = -2(y-20)$, $y(0) = 80$. Can the solution intersect the line $y = 20$? Justify your answer.

7. Find all solutions to the initial value problem

$$\frac{dx}{dt} = (x^3 - 5x + 2)(t^4 - t^2 - 2)\ ,\ x(5) = 2$$

8. What does Picard's Theorem tell us about solutions to the initial boundary problem

$$x\frac{dy}{dx} - y = 0,\ y(0) = 0?$$

Find infinitely many solutions to this IVP by sketching the direction field. What about the IVP $x\frac{dy}{dx} - y = 0$, $y(1) = 2$?

9. Prove that if $f(y)$ is a function whose first derivative is continuous, then any solution to the differential equation $\frac{dy}{dx} = f(y)$, is either strictly increasing, strictly decreasing, or constant.