Curvature

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Alice: Suppose you're driving a car and making a turn. How could you measure the rate at which the car is changing direction?

George: I suppose I could look at the compass on my dashboard and see how fast the needle is turning. Perhaps I could measure this rate in degrees/second.

Alice: So you could say something like "at this moment the car is turning at the rate of 10 degrees/sec to the right".

George: This measurement seems to depend upon two things - our speed and how the road is curving.

George: I wonder if we could measure how fast the road itself is turning. That measurement should be independent of how fast we were driving.

Alice: What units would we use to measure that?

George: We could say something like "the road turns at the rate of 2 degrees/ft at this point."

Alice: What does that mean?

George: That if I travelled one foot from that point, the car would turn approximately 2 degrees.

Alice: I see. That sounds suspiciously like a derivative.

George: Right. Then if I were driving my car at a speed of 40 feet/sec, when I reached that point in the road, the car would be turning at the rate of (40 ft/sec)(2 deg/ft) = 80 deg/sec.

Alice: Wait a minute. That sounds like you'd be turning pretty fast.

George: I guess that means that a road is turning pretty sharply if it turns at the rate of 2 degrees/ft.

Alice: O.K., but how do we actually go about computing the curvature of a road.

George: I'm not sure, but perphaps we could deal with the constant case first.

Alice: I guess a circle curves at a constant rate.

George: So if we travel once around a circle of radius R feet, then we will have turned through an angle of 360 degrees.

Alice: And we will have travelled a distance of $2\pi R$ feet.

George: So it seems like the curvature of a circle of radius R feet should be $360^\circ/(2\pi R)ft$.

Alice: I'm not sure I like using degrees to measure the turning angle. How about if we use radians instead.

George: Then we will have turned through an angle of 2π radians in travelling $2\pi R$ feet, so the curvature should be $2\pi/(2\pi R)$ rad/ft $= 1/R$ rad/ft.

Alice: That's a bit nicer. It also seems to make sense. The larger the radius the smaller the curvature.

George: So the units of curvature are rad/ft, or more generally rad/(unit length)?

Alice: That seems right. But remember radians themselves are dimensionless since a radian is a ratio of two lengths. So it seems like we could also say that the units of curvature are $1/ft$.

George: O.K. But can we calculate the curvature of anything else?

Alice: Well, the curvature of a straight line should be zero, since when you travel in a line your direction does not change.

George: How about the curvature of a general curve in two dimensions, like the curve $y = f(x)$? Oh wait, here comes our professor. Maybe he'll have something to say.

Professor: Sometimes it's easier to do the most general case. Our curve might not be given by a function. Why don't you try to find the curvature of the curve given parametrically by $\mathbf{r}(t) = < x(t), y(t) >$? Then you might be able to deduce the curvature of the function $y = f(x)$ as a special case. That's what we did when we calculated arc length, if you remember.

Alice: I can't say that I do. But how should we start?

Professor: You could think of t as representing time and first try to calculate the rate at which the direction of the particle is changing with respect to time.

George: Joe and I were just talking about that. Like I might say my car is turning at the rate of 30 degrees/sec.

Alice: But then how could we calculate the curvature of the road at that point?

George: If we knew our speed, then we could divide the turning rate by our speed.

Alice: So if we were turning at the rate of 30 degrees/sec while we were driving at a speed of 40 ft/sec, then the road would be turning at the rate of 0.75 degrees/ft at that point.

Professor: Precisely. Just don't forget to use radians instead of degrees. Well, good luck. I'm off to measure the dimensions of my bathtub.

George: Strange dude. But how can we measure the rate at which a particle is changing direction?

Alice: We probably should first find the direction the particle is moving.

George: That sounds like its related to velocity. It's easy to calculate the velocity of a particle if we know its position.

Alice: Yes, just take the derivative. So if $\mathbf{r}(t) =< x(t), y(t) >$, then $\mathbf{v}(t) =< x'(t), y'(t) >$.

George: Now what?

Alice: We still need to measure the direction the particle is moving at this instant.

George: That gets back to the idea of the compass in my car. The compass gives a direction as an angle measured clockwise from due north.

Alice: That's right. Choosing due north as the reference point seems a bit arbitrary. How about if we measure the angle the velocity vector makes with the positive x-axis?

George: O.K. So if ϕ is the angle the velocity vector makes with the positive x-axis, then

$$\phi = \tan^{-1}\left(\frac{y'(t)}{x'(t)}\right).$$

Alice: I suppose that's true only if ϕ is between $-\pi/2$ and $\pi/2$, but let's ignore that small technicality. Anyway, if we now differentiate both sides of this equation, we can determine the rate at which ϕ is changing. It looks like we get

$$\frac{d\phi}{dt} = \frac{1}{1+\left(\frac{y'(t)}{x'(t)}\right)^2} \cdot \frac{y''(t)x'(t) - y'(t)x''(t)}{\left(x'(t)\right)^2} \tag{1}$$

$$= \frac{y''(t)x'(t) - y'(t)x''(t)}{\left(x'(t)\right)^2 + \left(y'(t)\right)^2} \tag{2}$$

George: When is the drop date for this class?

Alice: It's not as bad as it looks. Remember, we've just calculated the turning rate in rad/sec. To get the curvature of the road we need to divide $d\phi/dt$ by the speed.

George: The speed is just the magnitude of the velocity vector, or $\sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}$. So then the curvature of the path is

$$\kappa(t) = \frac{d\phi/dt}{\mid \mathbf{v} \mid} = \frac{y''(t)x'(t) - y'(t)x''(t)}{\left(x'(t)\right)^2 + \left(y'(t)\right)^{3/2}}.$$

Alice: Just one thing. If we don't want to keep track of whether the road is bending to the right or left (which would depend on the direction we were moving), we should take the absolute value of the numerator, so that the curvature is never negative. Then we would have

$$\kappa(t) = \frac{\mid d\phi/dt \mid}{\mid \mathbf{v} \mid} = \frac{\mid y''(t)x'(t) - y'(t)x''(t) \mid}{\left(x'(t)\right)^2 + \left(y'(t)\right)^{3/2}}.$$

George: How about an example?

Alice: Let's take an ellipse like $\mathbf{r}(t) =< x(t), y(t) > = < a\cos t, b\sin t >$.

George: Then we could recover a circle of radius R as a special case by taking $a = b = R$.

Alice: Since $x(t) = a\cos t$, $x'(t) = -a\sin t$ and $x''(t) = -a\cos t$. Also since $y(t) = b\sin t$, $y'(t) = b\cos t$ and $y''(t) = -b\sin t$. Then we get

$$\kappa(t) = \frac{\mid ab \mid}{\left(a^2\sin^2 t + b^2\cos^2 t\right)^{3/2}} = \frac{ab}{\left(a^2\sin^2 t + b^2\cos^2 t\right)^{3/2}},$$

if we assume $a, b > 0$.

George: So what does this do for me?

Alice: Well, if $a = b = R$, then we have a circle and the curvature reduces to $1/R$. Also we might want to compute the curvature at the vertices of the ellipse.

George: What do you mean by the vertices of the ellipse?

Alice: The points where the curvature is locally a maximum or minimum. For the ellipse this just means the x and y intercepts.

George: We get an x-intercept when $t = 0$ and then

$$\kappa(0) = ab/b^3 = a/b^2.$$

Alice: And we get a y-intercept when $t = \pi/2$ and then

$$\kappa(\pi/2) = ab/a^3 = b/a^2.$$

George: Then the ratios of the two curvatures is $\kappa(0)/\kappa(\pi/2) = (a/b)^3$.

Alice: So if $a = 2$ and $b = 1$, then the curvature at the point $(2, 0)$ is eight times the curvature at the point $(0, 1)$. Quite a big difference. I wonder how we could see this?

George: Here comes the professor. I bet he can help.

Professor: Yes, yes, yes, I couldn't help overhearing your conversation. There are a couple of ways we could take a more geometric approach. One way would be to look at the osculating circle.

Alice: The what?

Professor: Circulum osculans in Latin, meaning kissing circle. Of all the circles that are tangent to a curve at a given point P, it is the one that best approximates the curve at P.

Alice: In what sense?

Professor: If you imagine two particles with the same speed, one moving along the curve and the other moving along the osculating circle, then they would both be turning at the same rate when they reached P.

George: So you just mean that the osculating circle and the curve have the same curvature at P.

Professor: Precisely.

Alice: But we know the curvature of a circle of radius R is $1/R$, so that must mean the radius of the osculating circle is $1/\kappa(t)$.

Professor: Precisely. Here have a look to see how the osculating circle varies around an ellipse.

See http://www.shoreline.edu/fkuczmarski/Math

George: As the osculating circle moves around the ellipse its radius changes quite dramatically.

Professor: Notice how the osculating circle crosses the ellipse at the point of tangency.

Alice: That's a lot different than the tangent line to a curve. Usually a tangent line does not cross the curve at the point of tangency, unless ...

George: Unless the point of tangency is an inflection point. I wonder why the osculating circle seems to always cross the ellipse at the point of tangency.

Professor: I'll let you think about that. But there are some points where the osculating circle does not cross the ellipse.

Alice: Those points seem to be at the vertices of the ellipse.

George: Hey, what is that red curve in the picture? It looks kind of like an astroid.

Professor: That's the evolute of the ellipse. It's the centers of all the osculating

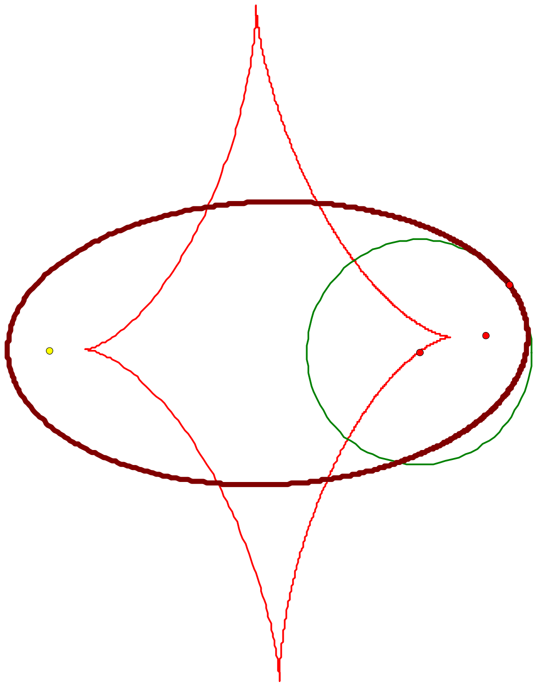

Figure 1: An ellipse with osculating circle

lating circles to the ellipse.

Alice: And it seems like the points where the osculating circles don't cross the ellipse correspond to cusp points on the evolute.

Professor: Excellent observation. You might want to think about why that might be true. But we're getting a bit sidetracked. I need to make two more points.

George: Actually, it's getting a bit late, maybe ...

Professor: Your method for computing curvature needs some modification for space curves.

Alice: Why is that?

Professor: You can no longer choose an arbitrary direction as your reference like you did when you chose the positive x-axis to measure the direction of motion of a particle.

George: Why not?

Professor: Suppose a particle is moving in the yz-plane in a circle centered at the origin. Then the velocity vector of the particle always makes a 90° angle with the positive x-axis, but certainly we wouldn't say that the particle is not changing direction.

Alice: So what should we do?

Professor: We could start out by parameterizing the given curve by arc length.

George: I've heard this term before, but I don't really understand what it means.

Professor: Let's take an example first. Are you familiar with I-5?

Alice: You mean the road? Sure. In Washington it runs more or less due north and south.

Professor: For our purposes, it would help to think of all the curves in the road as well. The exits parameterize I-5 by arc length.

George: How do you mean?

Professor: Each exit number measures the number of miles to the Washington-Oregon border as measured along the road.

George: Since the Shoreline exit is 176, that must mean Shoreline is 176 miles from the border.

Alice: But which part of the border?

George: It must be where I-5 crosses the border and the 176 miles must be measured along I-5, as if you were driving your car. Is that correct professor?

Professor: Yes, exactly. We can do the same thing with any curve. Just choose an arbitrary point on the curve as your origin and choose one of two directions along the curve to be the positive direction. Then just imagine marking the mile-markers (or foot markers, or whatever unit of measurement you choose) along the curve. It's like wrapping a number line on the curve.

Alice: But what exactly does it mean to parameterize the curve by arc length?

Professor: Well, if the curve were sitting in some coordinate system, parameterizing the curve by arc length would mean to express the coordinates of any point in terms of its signed distance to zero.

Alice: What do you mean by signed distance?

Professor: Remember, we choose one direction along the curve to be the positive direction. Points in the other direction from zero have negative signed distance from zero.

George: I think an example would help.

Professor: Do you know a parameterization of the circle of radius R centered at the origin?

George: I think we could use $\mathbf{r}(\theta) = < R\cos\theta, R\sin\theta >$.

Professor: How could we reparameterize this circle by arclength?

Alice: I'm not sure.

Professor: Well, we first need to choose a point on the circle where the arc length function s is equal to 0.

George: It seems like the point $(R, 0)$ would be a good choice.

Professor: That will work. We also need to choose a positive direction.

Alice: How about counter-clockwise?

Professor: Fine as well. Now suppose you have moved s units along the circle in the counter-clockwise direction from the point $(R, 0)$. What are your coordinates?

George: That's easy enough. The angle θ in our original parameterization would be s/R, so our coordinates would be $\mathbf{r}(s) = < R\cos(s/R), R\sin(s/R) >$
.

Professor: Correct. Now why were we talking about parameterizing a curve by arc length anyway?

Alice: You were telling us that our method for computing the curvature of a plane curve does not extend to three-dimenensions.

Professor: Yes, of course. So we begin by taking a curve in three dimensions and parameterizing it by arc length s. Another way to think of this parameterization is that s represents time and the particle is moving along the curve with a unit speed.

Alice: Like if I'm driving on I-5 at a speed of one mile/minute, then my distance s to the border is the same as my time of travel in minutes. Assuming I started driving at the border, that is.

Professor: Exactly.

George: I see how we parameterized a circle by arc length, but it seems like this might be difficult for a more complicated curve.

Professor: That's certainly true. We may not be able to write down a parameterization by arc length explicitly, but we can imagine that one exits.

Alice: And I can imagine I'm getting a 4.0 in this class.

Professor: If you wish. But now the main idea is this. If a particle is moving along a curve with unit speed, then the velocity vector always has unit length. So the acceleration vector, which measures the rate at which the velocity is changing, records only the rate at which the direction of the particle is changing, and we can use the acceleration vector to measure curvature.

George: How?

Professor: The length of the acceleration vector will measure the curvature.

Alice: I'm not sure that I understand why that is.

Professor: Suppose we have a plane curve parameterized by arc length. Then write the velocity vector as $\mathbf{T} = < \cos\phi, \sin\phi >$, where ϕ is the angle the velocity vector makes with the positive x-axis.

George: That angle ϕ is the same angle Joe and I were talking about earlier.

Professor: Yes. The curvature is then the rate at which ϕ is changing with respect to arc length, or symbolically, $\kappa = \mid d\phi/ds \mid$.

Alice: Earlier we computed $\mid d\phi/dt \mid$ and divided by the speed.

George: But the speed is just ds/dt.

Alice: Why is that?

George: Think of the I-5 example. Then s is just the distance from the border measured along I-5. It's the reading you would see on your odometer if you set it to zero when you were at the border.

Alice: I see. Then ds/dt is the rate at which your odometer is changing, which is your speed.

George: And the curvature is

$$\kappa = \mid \frac{d\phi/ds}{ds/dt} \mid = \mid d\phi/ds \mid .$$

Alice: So then

$$d\mathbf{T}/ds = \frac{d}{ds}(< \cos\phi, \sin\phi >) = \frac{d\phi}{ds} \; < -\sin\theta, \cos\theta > .$$

George: So that's the acceleration vector for a particle moving along a curve at unit speed. To get the curvature, we just take the magnitude of the acceleration vector, which is just $\mid d\phi/ds \mid$.

Alice: That agrees with how we defined curvature for plane curves. But what about the direction of the acceleration vector? Why don't we care about that?

Professor: The last equation tells us that the acceleration vector for a unit speed curve is always perpendicular to the curve, so we always know its direction. It's the length of the acceleration vector that gives us the curvature.

George: I'm still not sure I fully understand why this is true. I guess I'd like to be able to *see* why the length of the acceleration vector is the curvature.

Professor: Take a look at this.

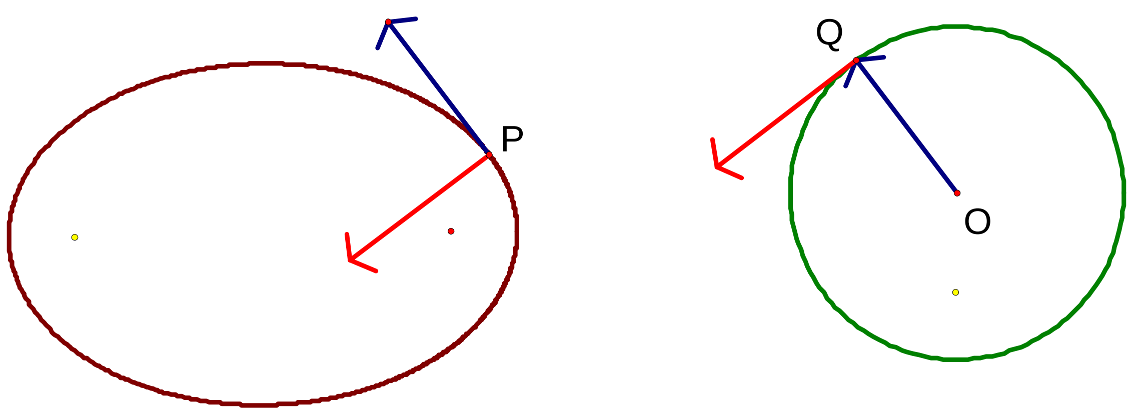

Figure 2: An ellipse with unit tangent and acceleration vectors

George: I'm watching point P move around the ellipse at unit speed and I'm watching the unit tangent vector move around the ellipse with P. I see that as the curvature of the ellipse increases, the length of the acceleration vector also increases. But what is that circle on the right?

Professor: We've just translated the unit tangent vectors so that they all emanate from a common point O. Then we can just watch the tangent vectors rotate as point P moves around the ellipse. The tips of the unit tangents at point Q now just move around a circle.

George: I see. It's kind of like when I drive my car and watch the GPS screen. There's an arrow on the screen that points in the direction I'm moving.

Professor: What's a GPS screen?

George: Never mind. But what is that vector tangent to the circle?

Professor: That's just the velocity vector of point Q.

George: I see. And it's also the acceleration vector of P.

Professor: Precisely. Now can you understand why the magnitude of this vector gives the curvature?

George: The curvature measures the rate at which the tangent vector is rotating. The magnitude of the velocity vector of Q is just the speed of Q. So you're saying that the speed of Q is the curvature of the ellipse at P.

Professor: Correct again.

George: I can see that the faster the velocity vector rotates, the faster Q will move, but I'm not quite sure why the speed of Q should equal the rate of rotation of the velocity vector.

Professor: Remember that Q is moving around a unit circle.

George: So its speed equals its rate of rotation about point O! So now it's clear. I see why the curvature at P equals the length of the acceleration vector as P moves around the ellipse with unit speed. But what if we have a curve in three dimensions?

Professor: Then Q will move on the surface of a unit sphere. Over a very short time interval, Q will be moving along a curve that's very close to a circular arc, and we can give the same argument. We'll look at an example in a bit.

Alice: But how does this help us compute the curvature? Especially if we don't know how to parameterize the curve by arc length?

Professor: Even if the curve is not parameterized by arc length, we can use the same idea of translating the tangent vectors so that their tails are all at a common point O. But now the tangent vectors are not necessarily of unit length, so their tips do not move on a circle if we're working in two dimensions, nor on a sphere if we're in three dimensions.

Alice: But I assume we at least start the same way. I mean, if we're given a parameterization of a curve, $\mathbf{r}(t)$, we compute the velocity vector $\mathbf{v}(t)$. Don't we?

Professor: Yes, of course. Let's assume we're in two dimensions, if that will help. The argument will work just as well in three.

Alice: Can we try to compute the rate at which the particle is changing direction, like we did we before? I mean the rate at which the particle is changing direction with respect to time? And then divide by the speed?

Professor: Precisely.

Alice: But how?

Professor: We need to review a bit first. Suppose a particle moves in a plane. How would we compute the rate at which the particle is turning about the origin?

Alice: That sounds like a related rates problem.

Professor: It is. To get started, suppose we know the distance, r, of the particle to the origin and the angle, θ, that the position vector of the particle makes with the positive x-axis at any time t.

Alice: That sounds like polar coordinates to me.

Professor: Precisely.

Alice: The position vector of the particle is $\mathbf{r} = r < \cos\theta, \sin\theta >$.

Professor: Good. What do you get if you differentiate position with respect to time?

Alice: Velocity. Then $\mathbf{v} = \frac{dr}{dt} < \cos\theta, \sin\theta > + \, r \, \frac{d\theta}{dt} < -\sin\theta, \cos\theta >$.

Professor: Now you've written the velocity vector as the sum of two vectors. Can you say anything about these components?

Alice: It looks like the first component, $\frac{dr}{dt} < \cos\theta, \sin\theta >$, is just a multiple of the position vector. So it points directly toward or away from the origin. And its magnitude, $\mid dr/dt \mid$, gives the absolute value of the rate at which the distance from the particle to the origin is changing.

Professor: Good. The first component is called the radial component of the velocity. What can you say about the second component?

George: It's perpendicular to the position vector.

Alice: And the magnitude of that component is $\mid r\frac{d\theta}{dt} \mid$.

Professor: Excellent again. This component is called the transverse component of the velocity.

George: This seems pretty intuitive. If the particle were moving directly away from or toward the origin, then the transverse component of the velocity vector would be zero.

Alice: At the other extreme, if the particle were moving in a circle about the origin, then the radial component of its velocity would be zero.

George: And its speed would just be $\mid r\frac{d\theta}{dt} \mid$, the magnitude of the transverse component. We've seen this formula before for the speed of a particle moving in a circle of radius r. In trigonometry class, we just wrote $v = r\omega$.

Alice: But how does this help us compute curvature?

Professor: Remember, we move the velocity vectors of the point P moving along the curve so that their tails are at a common point O. We first wish to determine the rate, with respect to time, at which the velocity vector is rotating at any instant.

Alice: But that's the rate at which the tip of the velocity vector, Q, is rotating about point O.

George: And this rate is the magnitude of the transverse component of the velocity vector of Q divided by its distance to O.

Alice: But the transverse component of the velocity of Q is $\mid \mathbf{v}_Q \mid \sin\phi$, where ϕ is the angle between the velocity vector $\mathbf{v}_Q$ and the position vector $\mathbf{r}_Q$.

George: And

$$\mid \mathbf{v}_Q \mid \ \sin\phi = \mid \mathbf{r}_Q \times \mathbf{v}_Q \mid / \mid \mathbf{r}_Q \mid .$$

Professor: Now remember that the position vector of Q is really the velocity vector, $\mathbf{v}$, of P, and the velocity of Q is really the acceleration, $\mathbf{a}$, of P.

Alice: So the transverse component of the velocity of Q is just

$$\mid \mathbf{r}_Q \times \mathbf{v}_Q \mid / \mid \mathbf{r}_Q \mid = \mid \mathbf{v} \times \mathbf{a} \mid / \mid \mathbf{v} \mid .$$

George: So the rate at which the velocity of P is changing direction with respect to time is just

$$\mid \mathbf{v} \times \mathbf{a} \mid / \mid \mathbf{v} \mid^2 .$$

Alice: And we divide by the speed of P again to get the curvature, so

$$\kappa = \mid \mathbf{v} \times \mathbf{a} \mid / \mid \mathbf{v} \mid^3 .$$

George: But what if we have a space curve?

Professor: The same argument works. We just work in the plane determined by the velocity and acceleration vectors of point P, or if you prefer, in the plane determined by the position and velocity vectors of Q.

Alice: Let's compute the curvature of a helix.

George: We can take the position vector to be

$$\mathbf{r}(t) = < r\cos t, r\sin t, kt > .$$

Alice: Then

$$\mathbf{v}(t) = < -r\sin t, r\cos t, k >,$$

and

$$\mathbf{a}(t) = < -r\cos t, -r\sin t, 0 > .$$

So

$$\kappa(t) = \mid \mathbf{v}(t) \times \mathbf{a}(t) \mid / \mid \mathbf{v}(t) \mid^3 = \frac{r}{r^2 + k^2}.$$

George: I don't follow that last step.

Alice: It's just algebra. Don't worry about it.

George: Can you convince me that it's correct?

Alice: Well, the units are the reciprocal of a length, which is correct. Also, if $k = 0$, then our helix is really a circle and the curvature becomes $1/r$, which agrees with the curvature of a circle we calculated before.

George: And if $r = 0$, the helix is just a vertical line with curvature zero, which is what our formula gives. But wait a minute. What happened to t in the formula for curvature?

Alice: It's gone. The curvature of a helix is constant.

George: I guess that's pretty intuitive. Since the curvature is constant, I wonder if we can compute the curvature of a helix without using the formula.

Alice: You mean like we did for a circle?

George: Yes, but how?

Professor: Take a look at the spherical image of the unit tangents to the helix. We do this just as before, by bringing the tails of the tangent vectors to a common point O.

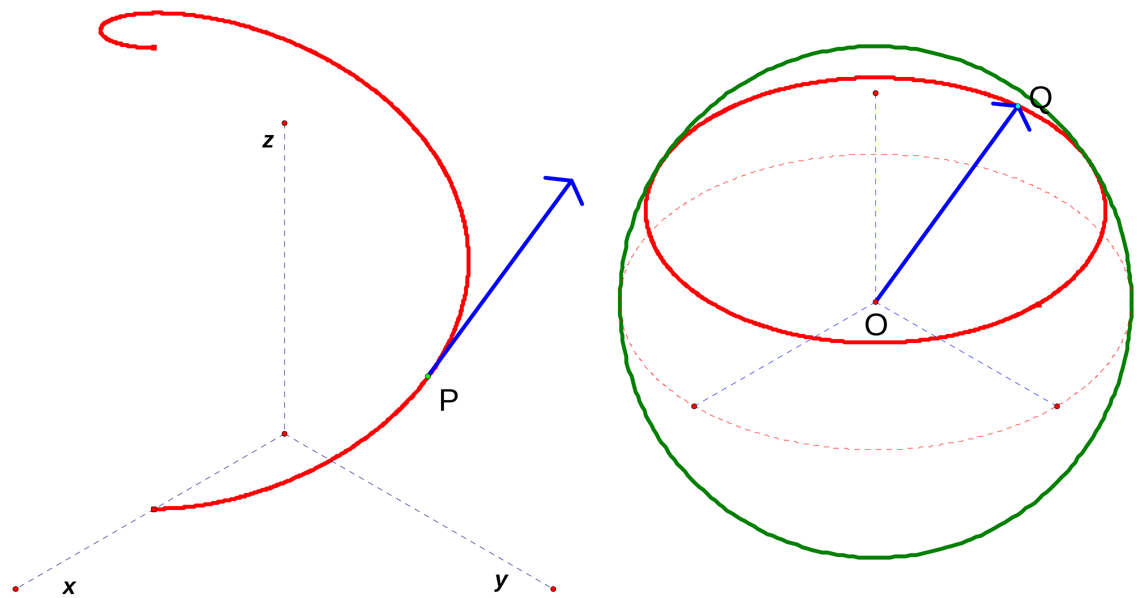

Figure 3: A helix with unit tangent vector and spherical image

George: It looks like the tips of the tangent vectors trace out a circle of latitude on the sphere.

Alice: That makes sense since the tangents to the helix make a constant angle with the xy-plane.

George: So let's see. When point P makes one complete revolution around the helix, the tangent vector at O returns to its starting position and therefore turns through 2π radians.

Alice: And point P travels $2\pi\sqrt{r^2 + k^2}$ units along the helix.

George: How do you know that?

Alice: The helix lies on a cylinder of radius r. If we cut this cylinder and unroll it, the helix unwraps to a line with slope k/r. I just computed the length of a line segment corresponding to the path P traces out as it makes one revolution around the helix.

George: So then the curvature would just be $2\pi/(2\pi\sqrt{r^2+k^2}) = 1/\sqrt{r^2+k^2}$.

Alice: But that doesn't match our result. Why?

Professor: Are you sure that the velocity vector turns through 2π radians as P rotates once around the helix?

George: Well, after one revolution the velocity vector is pointing in the same direction as it was before.

Alice: But the vector isn't rotating in a plane, so maybe it turns through less than 2π radians. It seems like the steeper the helix, the smaller the angle of rotation.

George: That sounds reasonable. But how can we calulate this angle?

Professor: What surface does the velocity vector at O sweep out as P makes one revolution about the helix?

Alice: Part of a cone!

George: What if we cut the cone open to get a sector of a circle like this?
Alice: Could it be that the central angle of the sector is equal to the angle through which the velocity vector rotates as P makes one revolution?

George: Perhaps. Instead of thinking about the total angle of rotation, it might help to think about the velocity vector turning through a small angle.

Alice: If point Q of the velocity vector moves through a small arc length on the circle of latitude, then the corresponding point Q' turns through the same small arc length on the cirumference of the sector, and the vectors $\vec{OQ}$ and $\vec{OQ'}$ will have turned through the same small angle.

George: I agree with that since the lengths of $\overline{OQ}$ and $\overline{OQ'}$ are both one.

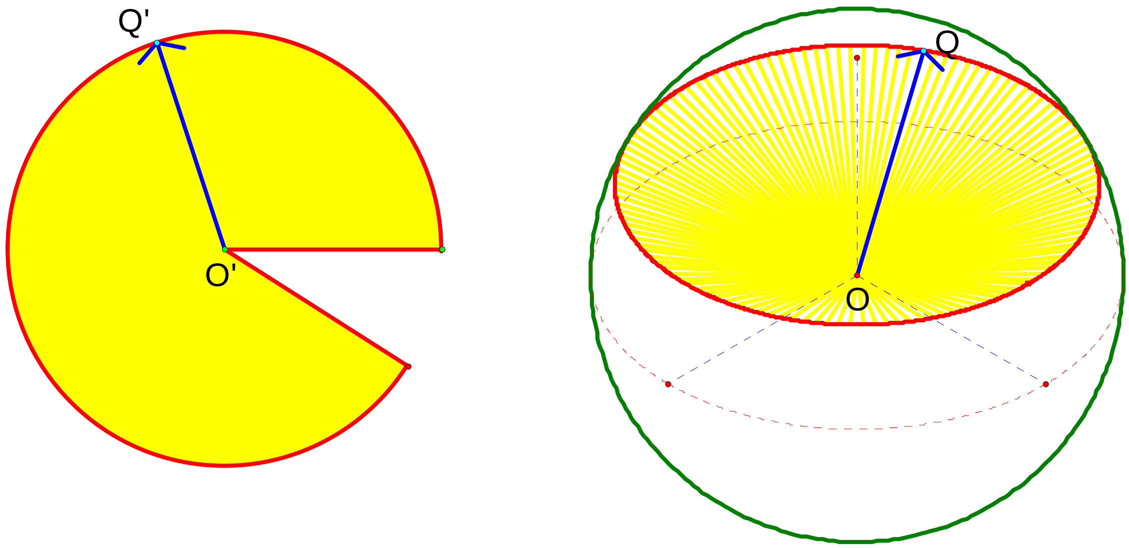

Figure 4: A sector of a cone

But how does this help?

Alice: Well, the total angle through which each of the vectors $\vec{OQ}$ and $\vec{OQ'}$ turn is just the sum of these small angles.

George: And since the arc length of the sector equals the arc length of the circle of latitude, the vector $\vec{OQ}$ turns through the central angle of the sector as P makes one revolution around the helix. Clever!

Alice: We should be able to calculate the curvature of the helix now.

George: The central angle of the sector is just the arc length of the sector since it has unit radius.

Alice: But the arc length of the sector is the circumference of the circle of latitude.

George: And the radius of the circle of latitude is $\cos\alpha$, where α is the angle that the tangent vectors to the helix make with the xy-plane. But didn't you say that $\tan \alpha = k/r$.

Alice: Yes, we can see that by unrolling the cylinder. So the circumference of the circle of latitude is

$$2\pi \cos \alpha = 2\pi r/\sqrt{r^2 + k^2}.$$

George: And this is the angle ω through which the velocity vector rotates as P makes one revolution about the helix.

Alice: But the point P travels $s = 2\pi\sqrt{r^2 + k^2}$ units as it makes one revolution.

George: So the curvature of the helix is $\omega/s = r/(r^2 + k^2)$.

Professor: Excellent. Now just two more minor points. Your expression for the curvature of a space curve,

$$\kappa(t) = \mid \mathbf{v}(\mathbf{t}) \times \mathbf{a}(\mathbf{t}) \mid / \mid \mathbf{v}(\mathbf{t}) \mid^3,$$

should reduce to the first formula you derived for the curvature of a plane curve,

$$\kappa = \frac{\mid d\phi/dt \mid}{\mid \mathbf{v} \mid} = \frac{\mid y''(t)x'(t) - y'(t)x''(t) \mid}{(x'(t))^2 + (y'(t))^{3/2}}.$$

Alice: I bet we can show this by parameterizing the plane curve in space by $\mathbf{r}(t) = < x(t), y(t), 0 >$.

Professor: Precisely. I'll let you work out the details. We also should be able to use this last expression for the curvature of a plane curve to find the curvature of a function $y = f(x)$. Do you see how?

Alice: We can always parameterize a function by letting $x(t) = t$ and $y(t) = f(t)$.

Professor: Precisely. I'll let you check, but then your formula for curvature reduces to

$$\kappa(x) = \frac{|\ f''(x)\ |}{(1 + (f'(x))^2)}.$$

Alice: So if the second derivative is zero, the curvature is also zero. That makes sense, since the curve is very straight when the second derivative is zero.

George: And if we're at an inflection point, then the curvature is zero. That's like if I'm driving my car and the car is transitioning from turning right to turning left, then at that instant the car is not turning at all.

Professor: What happens if the first derivative is zero?

Alice: Then the curvature is just the absolute value of the second derivative.

Professor: I'll let you think about why that's true. Perhaps that's enough for one day. Now what was I supposed to be doing?

George: I think you mentioned something about measuring a bathtub.

Professor: Yes, of course. Good day.

Alice: Maybe we should start the homework.

George: I usually wait until the day before the test to do that.

Alice: Good thinking.