Moviegoers arrive at rate $R(t) = 10\sin\left(\frac{\pi}{30}t\right)$ over a 30-minute period $0 \le t \le 30$. Since $\sin 0 = 0$, $\sin\frac{\pi}{2} = 1$, and $\sin \pi = 0$, their arrival rate starts at 0, increases until $R(t) = 10$ at $t = 15$, then decreases to zero at $t = 30$. Tickets can be sold at up to 5 tickets / min provided someone is there to buy one.

Two other times have special significance. These are the times at which $R(t)$ is equal to 5, the rate of possible ticket sales.

$$10\sin\left(\frac{\pi}{30}t\right) = 5$$

$$\sin\left(\frac{\pi}{30}t\right) = 0.5$$

$$\sin^{-1}\frac{1}{2} = \frac{\pi}{30}t \quad ; \quad \pi - \sin^{-1}\frac{1}{2} = \frac{\pi}{30}t$$

$$\frac{\pi}{6} = \frac{\pi}{30}t \qquad \pi - \frac{\pi}{6} = \frac{\pi}{30}t$$

$$t = \frac{\pi}{6}\cdot\frac{30}{\pi} = 5 \quad ; \quad \frac{5\pi}{6}\cdot\frac{30}{\pi} = t$$

$$t = \frac{150}{6} = 25$$

The first time, $t = 5$ represents the time at which the rate of arrivals, $R(t)$, first exceeds 5, the number which represents the ability of the box office to sell tickets. It is at $t = 5$ that the box office first falls behind and a line begins to form. The length of the line at time x minutes after 9:00 pm can be given by the integral

$$\int_5^x \left(10\sin\left(\frac{\pi}{30}\right) - 5\right)dt$$

The second significant time is $t = 25$ when the rate of arrival $R(t)$ dips below 5, and fewer people are arriving than being sold tickets. It is at this point that the line is longest and the box office begins to "catch up" with the demand for tickets

(a) We know that the line is longest when $t = 25$ because this is the first time $R(t)$ goes below 5, not including the first five minutes. The length of the line at $t = 25$ is given by:

$$L(25) = \int_5^{25} 10 \sin\left(\frac{\pi}{30} t\right) dt - \int_5^{25} 5\, dt$$

let $u = \frac{\pi}{30} t$

$\frac{du}{dt} = \frac{\pi}{30}$

$dt = \frac{30}{\pi} du$

$$= (10)\left(\frac{30}{\pi}\right) \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin u \, du - 5t \Big|_5^{25}$$

$$= -\frac{300}{\pi} \cos u \Big|_{\frac{\pi}{6}}^{\frac{5\pi}{6}} - (125 - 25)$$

$$= -\frac{300}{\pi}\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) - 100$$

$$= \frac{300\sqrt{3}}{\pi} - 100$$

or approximatly 65

(b) since a line starts forming at $t = 5$ and people arrive at $R(t) = 10 \sin\left(\frac{\pi}{30} t\right)$, if you arrive at $t = 15$ you are n^{th} in line starting with the first arrival at $t = 5$ (when a line starts to form.

$$n = \int_5^{15} 10 \sin\left(\frac{\pi}{30} t\right) dt$$

and this is equal to

$$\frac{1}{2} \int_5^{25} 10 \sin\left(\frac{\pi}{30} t\right) dt$$

by symmetry about $t = 15$.

(b) cont.

$$n = \frac{1}{2}\int_5^{25} 10\sin\left(\frac{\pi}{30}t\right)dt$$

$$= \frac{1}{2}\,\frac{300\sqrt{3}}{\pi} = \frac{150\sqrt{3}}{\pi}$$

with the box office selling 5 tickets / minute during this time period, it takes

$$\frac{150\sqrt{3}}{5\pi} \quad \text{or} \quad \frac{30\sqrt{3}}{\pi} \text{ minutes to clear the}$$

line ahead of you. You buy your ticket at

$$t = 5 + \frac{30\sqrt{3}}{\pi} \qquad \text{or approximately } 9{:}22$$

(c) The rate of tickets sold is equal to the arrival rate, unless there is a line in which case it is equal to 5/minute. Before $t=5$ the arrival rate is less than 5/minute, so the number of tickets sold from $t=0$ to $t=5$ is given by

$$k = 10\int_0^5 \sin\left(\frac{\pi}{30}t\right)dt$$

$$= -\frac{300}{\pi}\left[\cos\left(\frac{\pi}{30}t\right)\right]_0^5$$

$$= -\frac{300}{\pi}\left(\cos\frac{\pi}{6} - \cos 0\right)$$

$$k = -\frac{300}{\pi}\left(\frac{\sqrt{3}}{2} - 1\right) = \frac{300}{\pi}\left(\frac{2}{2} - \frac{\sqrt{3}}{2}\right) = \frac{300 - 150\sqrt{3}}{\pi}$$

$(12 < k < 13)$

(c) cont.

Once a line forms and the box office starts selling tickets at a rate of 5/minute, it will take

$\frac{150-k}{5}$ minutes to sell the remaining (150 − k) tickets

so the last ticket is sold at time t:

$$t = 5 + \frac{150}{5} - \frac{k}{5}$$

$$= 35 - \frac{300 - 150\sqrt{3}}{5\pi}$$

$$= 35 - \frac{60 - 30\sqrt{3}}{\pi}$$

or approximately $t = 32.44$

The last ticket is sold around 9:32

(since we know there are approximately 65 people in line at $t = 25$, and people are still arriving, and we know the maximum rate at which the box office can sell tickets is 5/min, we know there are still people in line at 9:32 and that the tickets are being sold at their maximum rate until the last ticket is sold.)

(d) If you purchased your ticket at 9:20 ($t = 20$) you were the 75 th person to arrive after 9:05 ($t=5$) when the line started to form, 15 minutes earlier. The number of arrivals after $t = 5$ is given by

$$n(x) = \int_5^x 10 \sin\left(\frac{\pi}{30} t\right) dt$$

Define x.

The time you arrived is given by x:

$$75 = 10 \int_5^x \sin\left(\frac{\pi}{30} t\right) dt$$

$$\frac{15}{2} = -\left(\frac{30}{\pi}\right) \cdot \left[\cos\left(\frac{\pi}{30} t\right) \right]_5^x$$

$$-\frac{\pi}{4} = \cos\left(\frac{\pi}{30} x\right) - \cos\frac{\pi}{6}$$

$$\frac{\sqrt{3}}{2} - \frac{\pi}{4} = \cos\left(\frac{\pi}{30} x\right)$$

$$\cos^{-1}\left(\frac{2\sqrt{3} - \pi}{4}\right) = \frac{\pi}{30} x$$

$$x = \cos^{-1}\left(\frac{2\sqrt{3} - \pi}{4}\right) \cdot \left(\frac{30}{\pi}\right)$$

or approximately $x = 14.25$

you arrived about 9:14 pm.