Math 124 Test #3 Name: KEY

Directions: Show all your work to receive credit. No credit for guessing. Answer all questions on another sheet of paper. Be organized and clear in your work. Answer all word problems with a concluding sentence. Clearly define any variables you introduce and include units in your definition. Include units in your answers when appropriate. Fully simplify all answers. For full credit, do not leave any trigonometric functions in your answers. For optimization problems, justify all assertions concerning minimum and maximum values. Also include a statement of your objective function with the appropriate domain and clearly indicate any constraints.

Give exact answers to the problems. An exact answer is something like π or $\ln 2$. Except for rational numbers, a calculator will not give you exact answers. While a non-graphing calculator is permitted, it will most likely not be of any help. Do not waste valuable time pressing buttons.

1. (6 points) The length of the edge of a cube is measured with a relative error of at most $\pm p\%$. Use the appropriate linear approximation to estimate the relative error in compute its volume. Assume p is small. No credit given for any other method.

2. (5 points) For what values of a and b does the function $f(x) = axe^{bx^2}$ have a global maximum at the point $(2, 6)$?

3. (5 points) Find the interval(s) on which the graph of $y = e^{5x} - e^x$ is concave down. Justify your answer.

4. (8 points) The minute hand of a clock is 8 cm long and the hour hand is 6 cm long. Find the rate at which the distance between the tips of the hands is changing at 9:00 pm.

5. (8 points) Of all right circular cylinders with a fixed volume V, determine the ratio of the height to the diameter of the cylinder with the smallest surface area.

6. (8 points) A wall of height a runs parallel to a tall building. The distance between the wall and the building is b feet. Determine the length of the shortest ladder that can reach from the ground over the wall and rest against the building.

1. The volume of a cube with edge length s is

$$V = s^3.$$

Thus,

$$\Delta V \approx \frac{dV}{ds} \cdot \Delta s = 3s^2 \cdot \Delta s, \quad \text{and}$$

$$\frac{\Delta V}{V} \approx \frac{3s^2 \Delta s}{s^3} = 3\left(\frac{\Delta s}{s}\right) = \pm 3p\%$$

The relative error in the volume is about $\pm 3p\%$.

2. If $f(x) = axe^{bx^2}$ has a local max. at $x = 2$, then $f'(2) = \left(ae^{bx^2} + 2abx^2e^{bx^2}\right)\Big|_{x=2}$

$$= a\left(e^{4b} + 8be^{4b}\right) = 0$$

and $a = 0$ or $b = -1/8$. If $a = 0$, then $f(x) \equiv 0$ and $f(2) \neq 6$. Thus, $b = -1/8$. Then

$$f(2) = 2ae^{-\frac{1}{2}} = 6, \quad \text{so} \quad a = 3\sqrt{e}.$$

We should also check that $(2,6)$ is really a global maximum.

3. The graph is concave down if

$$f''(x) = 25e^{5x} - e^{x} < 0,$$

or $$e^{x}\left(25e^{4x} - 1\right) < 0,$$

or $$e^{4x} < \frac{1}{25},$$

or $$x < \frac{1}{4}\ln\left(\frac{1}{25}\right)$$

4. Let θ be the angle between the hands and let c be the distance between the hands (in cm) at t hours past 9:00 pm. Then

$$c^2 = 100 - 96\cos\theta, \quad \text{so}$$

$$2c\,\frac{dc}{dt} = 96\sin\theta\,\frac{d\theta}{dt}.$$

At $t=0$, $c = 10$ and $\theta = \pi/2$. Also,

$$\frac{d\theta}{dt} = \frac{2\pi}{1} - \frac{\pi}{6} = \frac{11}{6}\pi \text{ rad/hr}.$$

At 9:00pm, the distance is changing at the rate of

$$\frac{dc}{dt} = \frac{1}{20}\cdot 96 \cdot \frac{11\pi}{6} = \frac{44\pi}{5} \text{ cm/hr}.$$

5. Let r be the radius,
h the height,
V the volume, and
S the surface area of the cylinder.

We wish to minimize $S = 2\pi r^2 + 2\pi r h$ subject to $V = \pi r^2 h$. Then we wish to minimize

$$S = 2\pi r^2 + \frac{2V}{r}, \qquad r > 0.$$

Then $\frac{dS}{dr} = 4\pi r - \frac{2V}{r^2} = 0 \Rightarrow$

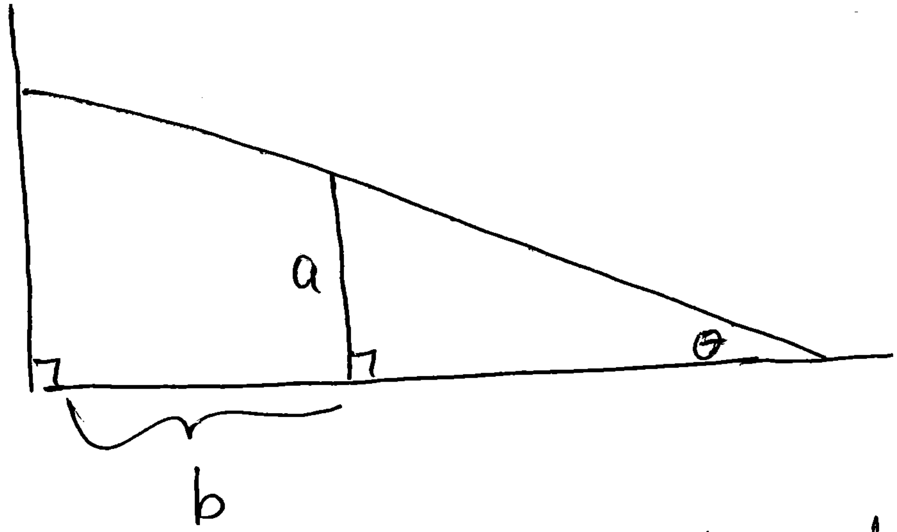
$V = 2\pi r^3$. But since $V = \pi r^2 h$,

$\pi r^2 h = 2\pi r^3$, and $h = 2r$;

and the height of the cylinder with the smallest volume is equal to its diameter. To check that we have actually found the minimum, note that

$$\frac{dS^2}{d^2r} = 4\pi + \frac{4V}{r^3} > 0 \text{ if } r > 0.$$

6.



Let θ be the angle the ladder makes with the ground and let L be the length of the ladder. We wish to minimize

$$L = \frac{a}{\sin\theta} + \frac{b}{\cos\theta}, \quad 0 < \theta < \pi/2.$$

then
$$\frac{dL}{d\theta} = \frac{-a\cdot\cos\theta}{\sin^2\theta} + \frac{b\sin\theta}{\cos^2\theta} = 0 \Rightarrow$$

$$\tan\theta = \sqrt[3]{a/b}.$$

Since $L \to \infty$ as $\theta \to 0^+$ or $\theta \to \pi/2^-$, this in fact gives the shortest ladder. Some algebra gives the length of the shortest ladder as

$$\left(a^{2/3} + b^{2/3}\right)^{3/2}.$$