Class Notes For Dynamics

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1 Sections 12.1-12.3: One Dimensional Motion

Mathematics Concept: The key idea is the Fundamental Theorem of Calculus: If f is a differentiable function, then $\int_a^b f'(t)\, dt = f(b) - f(a)$. In words, integrating the rate of change of a function over an interval gives the change in the function over that interval. It follows that

$$f(x) = f(a) + \int_a^x f'(t)\, dt.$$

Application: We can apply this to a particle moving in one-dimension. If $a(t)$, $v(t)$, and $s(t)$ represent the acceleration, velocity, and position of the particle at time t seconds, then

$$v(t) = v(0) + \int_0^t a(t)\, dt,$$

and

$$s(t) = s(0) + \int_0^t v(t)\, dt.$$

Note: Mathematicians frown on using the same variable for a limit of integration and the "dummy" variable. But since this is an engineering course, we will ignore this.

If the acceleration is constant (ie. $a(t) = a$), then

$$v(t) = v(0) + \int_0^t a\ dt = v(0) + at = v_0 + at$$

and

$$s(t) = s(0) + \int_0^t v(t)\ dt = s_0 + \int_0^t (v_0 + at)\ dt = s_0 + v_0 t + \frac{1}{2}at^2.$$

Mathematics Concept: The Average Value of a Function If f is a continuous function then the average value of f on the interval $[a, b]$ is given by

$$\frac{1}{b-a}\int_a^b f(t)\ dt.$$

Applications: The average velocity between $t = a$ and $t = b$ of a particle moving in one dimension is

$$\frac{1}{b-a}\int_a^b v(t)\ dt = \frac{\Delta s}{\Delta t}.$$

Since the speed of the particle is $\mid v(t) \mid$, its average speed between $t = a$ and $t = b$ is

$$\frac{1}{b-a}\int_a^b \mid v(t) \mid\ dt.$$

Question: What does the integral in the above expression represent?

Caution: The average velocity over the interval $[a, b]$ is almost never $\frac{1}{2}(v(a) + v(b))$. This is the case if the acceleration is constant. In this case it is also true that the average velocity equals the velocity at the midpoint of the time interval. This can be seen from the $v - t$ graph and from the $s - t$ graph.

One Final Note: We can eliminate t from the equations $v = ds/dt$ and $a = dv/dt$ to get $v\ dv = a\ ds$. Thus,

$$\int_0^v v\ dv = \int_0^s a\ ds,$$

or

$$\frac{1}{2}(v^2 - v_0^2) = \int_0^s a\ ds.$$

We will interpret this equation later using the Work-Kinetic Energy Theorem. But if the acceleration is constant we get

$$\frac{1}{2}(v^2 - v_0^2) = a(s - s_0).$$

Example 1: Starting from rest, a particle moving in a straight line has an acceleration given by $a(t) = 6t - 6$, where t is measured in seconds and $a(t)$ is measured in m/sec^2.

a. Determine the velocity at time $t = 6$.

b. Determine the change in position of the particle between times $t = 1$ and $t = 6$.

c. Determine the average velocity between times $t = 1$ and $t = 6$.

d. Determine the average speed between times $t = 1$ and $t = 6$.

e. Sketch a graph of the speed as a function of time.

Example 2: A particle moves along a straight line through a fluid medium such that its velocity (in m/sec) at time t seconds is given by $v(t) = 2(1 - e^{-0.3t})$.

a. Determine the displacement of the particle during the first three seconds of its motion.

b. Determine the speed of the particle at time $t = 3$ seconds.

c. Determine the average speed of the particle between times $t = 0$ and $t = 3$ seconds.

Example 3: The position (in meters) of a particle moving along a straight line at time t seconds is given by $s(t) = t^2 - 6t + 5$.
a. When does the particle change direction?
b. Determine the speed of the particle at time $t = 0$.
c. Determine the average velocity and average speed between times $t = 0$ and $t = 5$.

Example 4: The position (in meters) of a particle moving along a straight line at time t seconds is given by $s(t) = t^2 - 6t + 5$. Is the particle speeding up or slowing down at time $t = 1$?

Example 5: A particle is moving with a velocity of v_0 when $s = 0$ and $t = 0$. The acceleration of the particle is directly proportional to its velocity. If the velocity of the particle is $\frac{1}{2}v_0$ at time $t = 2$, determine the velocity and position in terms of time.

Example 6: A particle starts from rest at the origin and moves with an acceleration (in m/s^2) of $a = \frac{1}{3}\sqrt{s}$, where s is its position in meters. Determine the particle's velocity and position in terms of time.

Example 7: Particles A and B move along the same straight line such that the velocity of A at any point P is twice the velocity of B at point P. How do the accelerations of the particles compare at any point P?

Example 8: A plane starts from rest and has acceleration (in m/s^2) of $a = 25 - 0.1s$ after traveling s meters. Find the speed of the plane when it has traveled 100 feet. Do this algebraically and graphically.

2 Sections 12.4 - 12.7: The Motion of a Particle in Two or Three Dimensions

This should be mostly familiar. The only difference here is that position, velocity and acceleration are vectors. They were really vectors in one-dimensional motion as well since a scalar is a one-dimensional vector. The one topic which might cause some difficulty is the normal and tangential components of acceleration (pp. 49-51). The basic result is

$$\mathbf{a} = \frac{dv}{dt}\mathbf{u}_t + \frac{v^2}{\rho}\mathbf{u}_n, \tag{1}$$

where $\mathbf{u}_t$ is unit tangent vector to the path of the particle (pointing in the direction of motion) and $\mathbf{u}_n$ is the unit vector normal to the path (pointing to the concave side of the curve). Here v is the speed of the particle and ρ is the radius of curvature of the path. In Math 126 we wrote this as

$$\mathbf{a} = \frac{dv}{dt}\mathbf{u}_t + \kappa v^2\mathbf{u}_n,$$

where $\kappa = 1/\rho$ is the curvature of the path. Thus, the tangential component of acceleration is the time rate of change of speed. This is easy to prove:

$$\frac{dv}{dt} = \frac{d}{dt}(\mid \mathbf{v} \mid) = \frac{d}{dt}(\sqrt{\mathbf{v}\cdot\mathbf{v}}) = \mathbf{v}\cdot\mathbf{a}/\mid \mathbf{v} \mid .$$

The last expression is the scalar projection of $\mathbf{a}$ onto $\mathbf{v}$, or the tangential component of the acceleration vector. If $\frac{dv}{dt} > 0$ then the particle is speeding up and the angle between $\mathbf{a}$ and $\mathbf{v}$ is acute. If $\frac{dv}{dt} < 0$ then the particle is slowing down and the angle between $\mathbf{a}$ and $\mathbf{v}$ is obtuse. If $\frac{dv}{dt} = 0$ then the particle is maintaining a constant speed and the angle between $\mathbf{a}$ and $\mathbf{v}$ is a right angle.

The normal component of acceleration is a bit more complicated. As a special case, consider uniform circular motion, where

$$\mathbf{r} = r < \cos\omega t,\ \sin\omega t > .$$

We assume r and ω are positive constants, where ω is the angular speed (or turning rate) of the particle. Then

$$\mathbf{v} = r\omega < -\sin\omega t,\ \cos\omega t >,$$

and

$$\mathbf{a} = -r\omega^2 < \cos\omega t,\ \sin\omega t > = -\omega^2\mathbf{r}.$$

Note that the tangential component of accleration is zero since $\mathbf{v}\cdot\mathbf{a} = 0$. We should expect this since the particle is moving at a constant speed. Also,

$$v = \mid \mathbf{v} \mid = r\omega$$

and

$$a = \mid \mathbf{a} \mid = r\omega^2 = v^2/r.$$

Thus, in the special case of uniform circular motion, the acceleration vector points to the center of the circular path and has magnitude v^2/r. At any instant the path of a particle moving along some general path can be best approximated by a circle (the osculating circle) of radius ρ. "Best" means that the first and second derivatives of the path and circle agree at the point in question. Equation 1 says that the normal component of the particle's acceleration is v^2/ρ. Finally, since the vectors $\mathbf{a}_t$ and $\mathbf{a}_n$ are perpendicular, the magnitude of the acceleration is

$$a = \mid \mathbf{a} \mid = \sqrt{(dv/dt)^2 + (v^2/\rho)^2}.$$

Example 1: A particle moves counter-clockwise in a circle of radius 2 meters with an angular speed of 3 revolutions per second. Determine its speed and the magnitude of its acceleration.

Example 2: Determine the speed and the magnitude of the acceleration of a person in Seattle due to the rotation of the earth about its axis. Take the radius of the earth to be 3960 miles and the latitude of Seattle to be 47° N.

Example 3: A particle starts from rest and moves counter-clockwise in a circle of radius 6 meters, increasing its speed at a constant rate of 3 m/s^2. Determine the speed and magnitude of the acceleration of the particle at time $t = 2$ seconds.

Example 4: A particle moves along the parabola $y = \frac{1}{20}x^2$ in the direction of decreasing values of x. The particle's speed is 6 m/s and its speed is increasing at the rate of 2 m/s^2 when it reaches the point $(10, 5)$. Determine the direction and magnitude of the particle's acceleration at this instant. See Example 12.14 on page 53. Do this problem without using the equation for curvature given in the text.

Example 5: A particle moves counter-clockwise around a cirlce of radius 5 meters such that its angular speed (in rad/sec) at time t seconds is given by $\omega(t) = 2t$. Determine the magnitude of the acceleration of the particle at time $t = 2$.

3 Section 12.8: Cyclindrical Coordinates

We first use polar coordinates to describe two dimensional motion by writing the position vector as

$$\mathbf{r} = < r\cos\theta, r\sin\theta >,$$

where r and θ are both functions of t. Differentiating with respect to t gives

$$\begin{aligned} \mathbf{v} &= < \dot{r}\cos\theta - r\dot{\theta}\sin\theta,\ \dot{r}\sin\theta + r\dot{\theta}\cos\theta > \\ &= \dot{r} < \cos\theta,\ \sin\theta > + r\dot{\theta} < -\sin\theta,\ \cos\theta > \\ &= \dot{r}\mathbf{u}_r + \ r\dot{\theta}\mathbf{u}_\theta, \end{aligned} \tag{2}$$

where $\mathbf{u}_r = < \cos\theta, \sin\theta >$ is the unit vector pointing from the origin to the particle and $\mathbf{u}_\theta =< -\sin\theta, \cos\theta >$ is the unit vector perpendicular to $\mathbf{u}_r$ pointing in the direction of increasing values of θ. Note that $v_r = \dot{r} = dr/dt$ is the rate at which the distance between the particle and the origin is changing and is the component of the particle's velocity pointing directly away from the origin. Also, $v_\theta = r\dot{\theta}$ is the component of the particle's velocity in the direction of increasing θ. Since the vectors $\mathbf{u}_r$ and $\mathbf{u}_\theta$ are orthogonal, the speed of the particle is

$$v = \mid \mathbf{v} \mid = \sqrt{(\dot{r})^2 + (r\dot{\theta})^2}.$$

Differentiating equation 2 with respect to time and noting that $\frac{d}{d\theta}(\mathbf{u}_r) = \mathbf{u}_\theta$ and

$$\frac{d}{d\theta}(\mathbf{u}_\theta) = \frac{d}{d\theta} < -\sin\theta, \cos\theta > = \ - < \cos\theta, \sin\theta > = -\mathbf{u}_r$$

gives

$$\begin{aligned} \mathbf{a} &= \ddot{r}\mathbf{u}_r + \dot{r}\dot{\theta}\mathbf{u}_\theta + \dot{r}\dot{\theta}\mathbf{u}_\theta + r\ddot{\theta}\mathbf{u}_\theta - r(\dot{\theta})^2\mathbf{u}_r \\ &= (\ddot{r} - r(\dot{\theta})^2)\mathbf{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{u}_\theta \end{aligned}$$

Example 1: Simplify the above expressions for velocity and acceleration if the particle moves in a circle. Simplify the expressions further if the particle moves in a circle at a constant speed.

Example 2: (See Example 12.19, p. 69) A searchlight casts a spot of light along the face of a wall that is located 100 m from the searchlight. Find the speed and the magnitude of the acceleration at which the spot appears to travel across the wall at the instant $\theta = 45°$. Assume the light rotates at a constant rate of 4 rad/sec.

Example 3: A particle moves along the curve $r = 2\cos\theta$, where $\theta = 3t$ at time t seconds.

a. Show that the path of the particle is a circle of radius 1 centered at the point $(1, 0)$.

b. Show that the particle moves with a constant speed. Determine this speed.

c. Compare the values of $\dot{\theta}$ and ω.

d. Determine the angle between the vectors $\mathbf{v}$ and $\mathbf{r}$.

Example 4: Problem 12-153 on page 73.

4 Sections 16.3 - 16.4: Planar Kinematics of a Rigid Body: Rotation About a Fixed Axis

In this chapter we extend our study of kinematics from point masses to rigid bodies. We first study the case of a two-dimensional object rotating about a fixed axis. We assume the object is rotating in the xy-plane about the origin O. Let P be a point on the body and let θ denote the signed angle (clockwise angles are taken to be negative) measured from a fixed direction (usually the positive x-axis) to the vector $\vec{OP}$. Angular velocity, $\boldsymbol{\omega}$, is defined to be a vector

$$\boldsymbol{\omega} = \dot{\theta}\mathbf{k} = \omega\mathbf{k},$$

where $\omega = d\theta/dt$ is the signed turning rate. Note that the direction of $\boldsymbol{\omega}$ is given by the right-hand rule. The angular acceleration, $\boldsymbol{\alpha}$, is also a vector and is defined to be the time derivative of $\boldsymbol{\omega}$. Thus,

$$\boldsymbol{\alpha} = d\boldsymbol{\omega}/dt = \ddot{\theta}\mathbf{k} = \alpha\mathbf{k},$$

where $\alpha = d\omega/dt$.

Question: How can you use the vectors $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ to tell if the turning rate is increasing or decreasing?

Any point P not on the axis of a rotating body moves in a circular path. If r denotes the radius of this path, then the speed of P is ωr. Its velocity is given by

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}_P, \tag{3}$$

where $\mathbf{r}_P$ is the position of P with respect to any point Q on the axis of rotation. The above cross-product is independent of the position of Q as long as Q lies on the axis of rotation.

Since the object might not be rotating at a constant rate, the tangential component of acceleration is not necessarily zero. But remembering that the tangential component of acceleration is the time rate of change of speed and assuming that $\omega > 0$, we can compute a_t as

$$a_t = \frac{dv}{dt} = \frac{d}{dt}(\omega r) = \alpha r.$$

The normal component of acceleration is the same as in the case of uniform circular motion, namely

$$a_n = \omega^2 r.$$

Differentiating equation 3 with respect to time gives

$$\begin{aligned}\boldsymbol{\alpha} &= \boldsymbol{\alpha} \times \mathbf{r}_P + \boldsymbol{\omega} \times \mathbf{v}_P \\ &= \boldsymbol{\alpha} \times \mathbf{r}_P + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_P) \\ &= \boldsymbol{\alpha} \times \mathbf{r}_P - \boldsymbol{\omega}^2 \mathbf{r}_P \\ &= \alpha \mathbf{k} \times \mathbf{r}_P - \boldsymbol{\omega}^2 \mathbf{r}_P \qquad (4)\end{aligned}$$

Example 1: The angular velocity (in rad/sec) of a disk of radius 0.5 meters at time t seconds is given by $\omega = 8t^2 + 2$. Determine the speed and the magnitude of the acceleration of a point on the edge of the disk at time $t = 0.5$ seconds.

Example 2: Problem 16-7, p. 313.

Example 3: Problem 16-13, p. 314.

Example 4: A wheel of radius r rolls on the ground without slipping with angular velocity ω and angular acceleration α. Determine the velocity and acceleration of the center of the wheel.

Example 5: Problem 16-36, p. 323

Example 6: A ladder of length L slides down a tall wall as the base of the ladder is dragged along the ground. At the instant the ladder makes a 45° with the ground, the base of the ladder is moving with a speed of v m/sec and has an acceleration of a m/s^2. Determine

i. the angular velocity and angular acceleration of the ladder

ii. the speed of the top of the ladder
iii. the acceleration of the top of the ladder

at this instant.

Example 7: Problem 16-44, p. 325

5 Section 16.5-16.6: General Motion in Two Dimensions: Velocity

Imagine a square with its center initially at the origin and its sides parallel to the coordinate axes. If the square does not rotate and the center of the square moves in a straight line (rectilinear translation), then each point of the square moves in a straight line parallel to the path of the center. Thus, this is no different from the motion of a point, which we studied in Chapter 12. Imagine instead that the center of the square moved along some arbitrary curve $\mathcal{C}$, while the square did not rotate (curvilinear translation). Then each point of the square moves along a translated copy of the curve $\mathcal{C}$, and to understand the motion of the square it suffices to understand the motion of its center. A third possibility is that the square rotates about a fixed point. We studied this rotational motion about a fixed axis in the previous section. We now study the most general type of motion in two dimensions where the square translates and rotates simultaneously.

To study the motion of an arbitrary point B of the square it suffices to study the motion of one particular point of the square A and the motion of B relative to A. If O is the origin and $\mathbf{r}_A = \vec{OA}$, $\mathbf{r}_B = \vec{OB}$, and $\mathbf{r}_{B/A} = \vec{AB}$, then

$$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A}.$$

Differentiating both sides of this equation with respect to time gives

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}, \tag{5}$$

where $\mathbf{v}_{B/A}$ denotes the velocity of B relative to A. But since the distance between A and B remains constant, B moves in a circular path relative to A and we know from the last section that

$$\mathbf{v}_{B/A} = \boldsymbol{\omega} \times \mathbf{r}_{B/A},$$

where ω is the angular velocity of the square.

Example 1: See Example 16.6 on page 331.

Example 2: See Example 16.8 on page 333.

Example 3: See Example 16.9 on page 334.

It is a surprising theorem that at any instant the motion of a body in two dimensions can be described as a rotation about some point provided that the body is not in pure translational motion. The center of rotation is called the instantaneous center (IC) of rotation. Since the IC has zero velocity, the velocity of a point B is simply the velocity of B relative to the IC. If the directions of the velocities of points A and B of the body are known, then the IC may be constructed as the intersection point of the lines through A and B perpendicular to $\mathbf{v}_A$ and $\mathbf{v}_B$, respectively.

Example 4: A wheel of radius r rolls on a flat surface without slipping with a constant angular speed of ω rad/sec. Determine the speeds of the center of the wheel and the top point of the wheel. Determine the velocities of other points on the circumference of the wheel.

Example 5: Redo example 16.8 on page 333.

Example 6: A wheel of radius R rolls on the circumference of a stationary wheel of radius r without slipping. If the line segment between the two centers of the wheels rotates at the rate of ω rad/sec, determine the angular speed of the rolling wheel. (See problem 16-75)

Example 7: Problem 16-65, p. 338. Note that the picture is poorly drawn. If $\theta = 60°$, then the bar BC should point directly to the center of disk D.

Find the instantaneous center of bar BC.

Example 8: Example 16.11 on page 345. Do this problem two ways. First by "related rates", then by finding the IC of point D.

Example 9: Example 16.12 on page 346. Do this problem two ways; first by using the reference frame of the bottom plate, then by finding the IC of the cylinder.

6 Section 16.7: Relative Motion Analysis: Acceleration

Differentiating equation 5 with respect to time gives

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}, \tag{6}$$

where $\mathbf{a}_{B/A}$ is the acceleration of B relative to A. For a rigid body in general plane motion, B moves in a circular path about A with angular velocity $\boldsymbol{\omega}$ and angular acceleration $\boldsymbol{\alpha}$. It follows from equation 4 that

$$\mathbf{a}_{B/A} = \alpha\mathbf{k} \times \mathbf{r}_{B/A} - \omega^2\mathbf{r}_{B/A},$$

where $\mathbf{r}_{B/A}$ is the position of B relative to A. Thus,

$$\mathbf{a}_B = \mathbf{a}_A + \alpha\mathbf{k} \times \mathbf{r}_{B/A} - \omega^2\mathbf{r}_{B/A}. \tag{7}$$

Example 1: Example 16.15 on page 356.

Example 2: Example 16.13 on page 354. Do two ways.

Example 3: Example 16.17 on page 358. Compare this with the previous example.

Sections 13.3-13.4: Equations of Motion for a System of Particles

Suppose our system consists of n discrete particles with masses m_i and total mass $m = \sum_{i=1}^{n} m_i$. Let $\mathbf{F}_i$ represent the net external force acting on the i th particle and let f_i represent the net internal force acting on the i th particle. Then Newton's second law gives

$$\mathbf{F}_i + \mathbf{f}_i = m_i \mathbf{a}_i.$$

Summing these n equations gives

$$\sum_{i=1}^{n} \mathbf{F}_i + \sum_{i=1}^{n} \mathbf{f}_i = \sum_{i=1}^{n} m_i \mathbf{a}_i.$$

But by Newton's third law the internal forces come in equal (in magnitude) and opposite (in direction) pairs and so $\sum_{i=1}^{n} \mathbf{f}_i = 0$. Thus,

$$\sum_{i=1}^{n} \mathbf{F}_i = \sum_{i=1}^{n} m_i \mathbf{a}_i.$$

Now

$$\begin{aligned} \sum_{i=1}^{n} m_i \mathbf{a}_i &= \frac{d^2}{dt^2}(\sum_{i=1}^{n} m_i \mathbf{r}_i) \qquad (8) \\ &= m \frac{d^2}{dt^2}(\frac{1}{m}\sum_{i=1}^{n} m_i \mathbf{r}_i) \\ &= m \frac{d^2}{dt^2}(\mathbf{r}_G) \\ &= m \mathbf{a}_G, \end{aligned}$$

where the vectors $\mathbf{r}_G$ and $\mathbf{a}_G$ denote the position and acceleration of the center of mass of the system, respectively. Thus,

$$\sum_{i=1}^{n} \mathbf{F}_i = m \mathbf{a}_G.$$

In the case of a continuous mass distribution, the above sums need to be replaced with integrals, but the result is the same: the net external force acting on the system is the product of the mass of the system and the acceleration of the center of mass.

Example 1: Example 13.1 page 112.

Example 2: Example 13.2, page 113.

Example 3: Example 13.3, page 114.

7 Section 13.5: Equations of Motion: Normal and Tangential Components

Example 1: Example 13.6, page 129.

Example 2: Example 13.8, page 131.

Example 3: Example 13.9, page 132.

Example 4: Problems 13-68, 13-60, 13-70

8 Section 13.6: Equations of Motion: Cylindrical Coordinates

Example 1: Example 13.10, page 141.

Example 2: Example 13.11, page 142.

Example 3: Example 13.12, page 143.

Example 4: Problem 13-88, page 144.

Example 5: Probme 13-110, page 149.

9 Section 17.1: Moment of Inertia

The moment of inertia I of a two or three dimensional body about an axis L is defined to be

$$I = \int r^2 \, dm$$

where dm is the mass element and r is the (perpendicular) distance from the mass element to the axis L. The integral is taken over the entire body. If the body has uniform density ρ, then $dm = \rho \, dV$ and the above integral becomes

$$I = \int r^2 \rho \, dV = \rho \int r^2 \, dV.$$

The SI units of I are $kg \cdot m^2$.

Example 1: Determine the moment of inertia of a cylinder of uniform density with mass m kilograms and radius r meters about its axis of symmetry.

Example 2: Determine the moment of inertia of a bar of uniform density with mass m kilograms and length l meters about an axis passing through the midpoint of the bar and perpendicular to the bar.

The Parallel Axis Theorem: Given a body and a fixed direction in space $\mathbf{v}$, the parallel axis theorem relates the moment of inertia, I, of the body about an axis L parallel to $\mathbf{v}$ to the moment of inertia, I_G, of the body about the axis through G (the center of mass of the body) parallel to $\mathbf{v}$. Specifically,

$$I = I_G + md^2$$

where m is the mass of the body and d is the distance between the two parallel axes. Thus, of all axes in space having a given direction, the axis which passes through the center of mass gives the smallest moment of inertia.

Example 3: Determine the moment of inertia of a bar of uniform density with mass m kilograms and length l meters about an axis passing through the end of the bar and perpendicular to the bar.

Example 4: Determine the moment of inertia of a solid sphere of uniform density with mass m kilograms and radius r meters about an axis passing through the center of the sphere.

10 Sections 17.2 - 17.5: General Plane Motion

Consider a two-dimensional object moving in a fixed plane, which we take to be the plane of our paper. The instantaneous axis of rotation has a fixed direction perpendicular to the plane of motion. The angular velocity and angular acceleration vectors are parallel to the axis of rotation and thus point in a fixed direction. We can therefore represent these three-dimensional vectors as scalars, where by convention a positive value for α or ω represents a vector pointing out of the page, and a negative value represents a vector pointing into the page.

As before, suppose our system consists of n discrete particles with masses m_i and total mass $m = \sum_{i=1}^{n} m_i$. Let $\mathbf{F}_i$ represent the net external force acting on the i th particle and let f_i represent the net internal force acting on the i th particle. Then Newton's second law gives

$$\mathbf{F}_i + \mathbf{f}_i = m_i \mathbf{a}_i.$$

Let $\mathbf{r}_i$ denote the position of the i th particle with respect to a fixed point P. Then

$$\mathbf{r}_i \times \mathbf{F}_i + \mathbf{r}_i \times \mathbf{f}_i = \mathbf{r}_i \times m_i \mathbf{a}_i.$$

Summing these n equations gives

$$\sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{F}_i + \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{f}_i = \sum_{i=1}^{n} \mathbf{r}_i \times m_i \mathbf{a}_i.$$

Since the internal forces come in equal and opposite pairs, $\sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{f}_i = \mathbf{0}$ (Note: This takes some thought. It is not as trivial as it might seem). Thus,

$$\sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{F}_i = \sum_{i=1}^{n} \mathbf{r}_i \times m_i \mathbf{a}_i. \tag{9}$$

Now $\sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{F}_i = \sum_{i=1}^{n} (\mathbf{M}_P)_i = \left(\sum_{i=1}^{n} (M_P)_i\right) \mathbf{k}$ is the sum of the moments of the external forces acting on the system about point P. We now rewrite the right hand side of equation 9. By equation 7,

$$\mathbf{a}_i = \mathbf{a}_P + \alpha \mathbf{k} \times \mathbf{r}_i - \omega^2 \mathbf{r}_i.$$

Thus,

$$\begin{aligned} \sum_{i=1}^{n} \mathbf{r}_i \times m_i \mathbf{a}_i &= \sum_{i=1}^{n} m_i(\mathbf{r}_i \times \mathbf{a}_P) + \sum_{i=1}^{n} m_i(\mathbf{r}_i \times (\alpha \mathbf{k} \times \mathbf{r}_i)) - \omega^2 \sum_{i=1}^{n} m_i(\mathbf{r}_i \times \mathbf{r}_i). \\ &= \left(\sum_{i=1}^{n} m_i \mathbf{r}_i\right) \times \mathbf{a}_P + \alpha \sum_{i=1}^{n} m_i(\mathbf{r}_i \times (\mathbf{k} \times \mathbf{r}_i)) \\ &= m\mathbf{r}_G \times \mathbf{a}_P + \alpha \sum_{i=1}^{n} m_i r_i^2 \mathbf{k} \\ &= \mathbf{r}_G \times m\mathbf{a}_P + (I_P \alpha)\mathbf{k} \end{aligned}$$

Thus, equation 9 becomes

$$\left(\sum_{i=1}^{n} (M_P)_i\right) \mathbf{k} = \mathbf{r}_G \times m\mathbf{a}_P + (I_P \alpha)\mathbf{k}. \tag{10}$$

We consider some special cases:

1. If P coincides with the center of mass G, then $\mathbf{r}_P = \mathbf{0}$ and

$$\sum_{i=1}^{n} (M_G)_i = I_G \alpha.$$

2. If the object is rotating about a fixed axis through P then $\mathbf{v}_P = \mathbf{a}_P = \mathbf{0}$ and

$$\sum_{i=1}^{n} (M_P)_i = I_P \alpha.$$

We now rewrite equation 10. By the parallel-axis theorem,

$$I_P = I_G + mr_G^2,$$

and by equation 7

$$\mathbf{a}_P = \mathbf{a}_G + \mathbf{a}_{P/G} = \mathbf{a}_G + \alpha\mathbf{k} \times (-\mathbf{r}_G) + \omega^2\mathbf{r}_G.$$

Thus,

$$\begin{aligned} \mathbf{r}_G \times m\mathbf{a}_P &= \mathbf{r}_G \times m\mathbf{a}_G - m\alpha\mathbf{r}_G \times (\mathbf{k} \times \mathbf{r}_G) + m\omega^2\mathbf{r}_G \times \mathbf{r}_G \\ &= \mathbf{r}_G \times m\mathbf{a}_G - m\alpha r_G^2\mathbf{k} \end{aligned}$$

Finally, equation 10 becomes

$$\begin{aligned} \left(\sum_{i=1}^{n}(M_P)_i\right)\mathbf{k} &= \mathbf{r}_G \times m\mathbf{a}_P + (I_P\alpha)\mathbf{k} \\ &= \mathbf{r}_G \times m\mathbf{a}_G - m\alpha r_G^2\mathbf{k} + (I_G + mr_G^2)\alpha\mathbf{k} \\ &= \mathbf{r}_G \times m\mathbf{a}_G + I_G\alpha\mathbf{k} \end{aligned}$$

This is equation (17-9) on page 401.

Example 1: A uniform 50-kg crate measuring $1m \times 1m \times 1m$ rests on a horizontal surface with $\mu_k = 0.2$. Determine the acceleration of the crate if a horizontal force of 600 N is applied to the crate 0.8 meters from the bottom of the crate. (Example 17.7, page 407).

Example 2: Problem 17-26, page 409.

Example 3: A rectangular box of uniform density having width w and height h is placed on a ramp that makes an angle of θ with the ground. The coefficient of static friction between the box and the ramp is μ_s. Show that if $\mu_s < \tan\theta$ then the box will slide down the ramp, but if $\mu_s > \tan\theta$ and $h > w/\mu_s$, then the box will tip over and not slide. What happens if $\mu_s > \tan\theta$ and $h < w/\mu_s$?

Example 4: Problem 17-47, page 413.

Example 5: A uniform bar of mass m kg and length l meters is pin supported at one end and hangs vertically in equilibrium. At what point on the bar may a horizontal force be applied so that the horizontal component of the reaction at the pin is zero? This is called the center of percussion of the bar (see problems 17-65 and 17-67).

Example 6: A uniform bar of mass m kg and length l meters is pin supported at one end. The bar is held horizontally and then released (see figure on page 418).

i. Determine the initial acceleration of the free end of the rod and the force exerted by the pin on the rod.

ii. Determine the angular velocity and angular acceleration of the rod after it has turned through an angle of θ radians and determine the force exerted by the pin on the rod at this instant.

Example 7: a) A long strip of paper is wrapped into a cylindrical roll of radius r meters and mass m kilograms. The free end of the paper is held and the roll is dropped. Determine the initial tension in the paper and the angular acceleration of the roll.
b) A long strip of paper is wrapped into two cylindrical rolls, each having a radius of r meters and a mass of m kilograms. The top roll is supported about its center but the bottom roll is not centrally supported. If the bottom roll is brought into contact with the top roll and released from rest, determine the initial tension in the paper between the rolls and the angular acceleration of each roll (see problem 17-104).

Example 8: A bowling ball is thrown without spin with an initial speed of v_0 m/sec. Determine the distance the ball travels before it starts to roll without slipping. Compare with problem 17-108.

Example 9: A cord of neglible mass is wrapped around a uniform cylindrical disk of radius r and mass m. If the cord exits the cylinder at its top, determine the maximum force that can be applied to the cord so that the disk rolls without slipping. Assume the coefficient of static friction is μ_s.

Ans: $3\mu_s mg$

Example 10: (Very difficult) The top of a ladder slides down a smooth wall while the base slides along a smooth floor. At the instant the ladder makes an angle of θ with the wall, the ladder has an angular velocity of ω rad/sec. If the length of the ladder is l meters, determine the angular acceleration of the ladder at this time. *Ans:* $\frac{3}{2}\frac{g}{l}\sin\theta$

11 Sections 14.1-14.3: Work and Energy

If a constant force $\mathbf{F}$ acts on a particle and the particle undergoes a straight-line displacement of $\mathbf{r}$, then the work done by the force on the particle is the scalar $U = \mathbf{F}\cdot\mathbf{r} = Fr\cos\theta$, where θ is the angle between the vectors $\mathbf{F}$ and $\mathbf{r}$. The SI units of work are joules, where 1 joule equals 1 $N \cdot m$. Thus, only the component of the force in the direction of the displacement does work on the particle. If the angle between $\mathbf{F}$ and $\mathbf{r}$ is acute (obtuse), then the force does positive (negative) work on the particle. If the angle is 90°, then the force does no work on the particle.

If the force is not constant or the motion of the particle is not in a straight line, then the work done by the force on the particle as the particle moves from position $\mathbf{r}_1$ to position $\mathbf{r}_2$ is given by

$$U = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}\cdot d\mathbf{r} = \int_{s_1}^{s_2} F\cos\theta\, ds = \int_{s_1}^{s_2} F_t\, ds, \tag{11}$$

where s is the arc length measured along the path of a particle from some arbitrary starting point, and F_t is the tangential component of the force. Two special cases deserve mentioning.

1. If the force $\mathbf{F}$ is constant (in direction and magnitude), then equation 11 becomes

$$U = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}\cdot \mathbf{dr} = \mathbf{F}\cdot \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{dr} = \mathbf{F}\cdot(\mathbf{r}_2 - \mathbf{r}_1),$$

and the work done by the force is independent of the path of the particle. An important example is the work done by the force of gravity (or weight $\mathbf{W}$) on a particle near the surface of the earth. Here, the work done by the force of gravity on a particle of weight $\mathbf{W}$ as it moves along some path from $\mathbf{r}_1$ to $\mathbf{r}_2$ is

$$U = \mathbf{W} \cdot (\mathbf{r}_2 - \mathbf{r}_1) = W \Delta h,$$

where Δh is the change in the height of the particle.

2. If the force $\mathbf{F}$ has a constant magnitude F and is always in the direction of the velocity of the particle, then equation 11 becomes

$$U = \int_{s_1}^{s_2} F \cos\theta \, ds = F \int_{s_1}^{s_2} ds = F(s_2 - s_1).$$

The Work-Kinetic Energy Theorem: This theorem states that the work done by the net external force $\mathbf{F}$ acting on a particle is equal to the change in kinetic energy of the particle. The proof is straightforward if we recall that the tangential component of acceleration is the time rate of change of speed of the particle, so that

$$F_t = ma_t = m\frac{dv}{dt} = m\frac{dv}{ds}\frac{ds}{dt} = mv\frac{dv}{ds}.$$

Thus,

$$U = \int_{s_1}^{s_2} F_t \, ds = \int_{s_1}^{s_2} mv\frac{dv}{ds} \, ds = \int_{v_1}^{v_2} mv \, dv = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2.$$

Example 1: Problem 14-2, page 179.

Example 2: Problem 14-8, page 180.

Example 3: Problem 14-17, page 182.

Example 4: A ball of mass m is attached to one end of a string of lenght r. The other end of the string is held fixed while the ball is held horizontally and released. Determine the tension in the string when the string is vertical.

Example 5: Problem 14-28, page 185.