

FINDING VOLUMES BY INTEGRATION

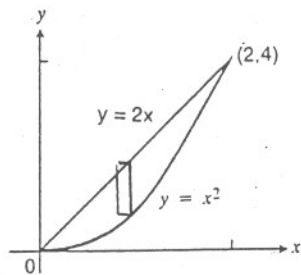
SHELL METHOD



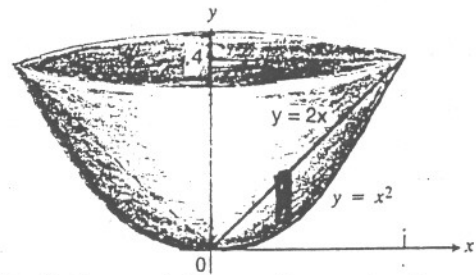
Overview

There are two commonly used ways to compute the volume of a solid -- the **Disk Method** and the **Shell Method**. Both involve slicing the volume into small pieces, finding the volume of a typical piece, and then adding up all the little pieces to form a Riemann Sum.

Consider the region in the first quadrant bounded by the curves $y = 2x$ and $y = x^2$. By solving these equations simultaneously we can see that the two curves intersect at the points $(0,0)$ and $(2,4)$. Suppose we wish to compute the volume of the solid formed when this region is rotated about the Y-axis.



Region to be rotated

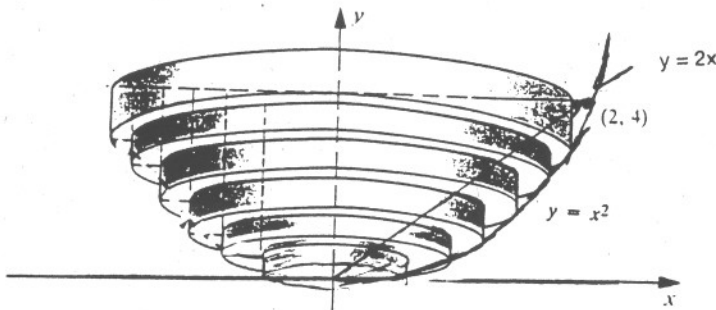


Bowl-shaped figure formed when region is rotated about the Y-axis

This handout sheet will only discuss the Shell Method, but there is another handout which computes the volume of this same solid by using the Disk Method.

Computing the Volume of One Shell

We will now compute the volume of this same "bowl". Imagine that the bowl is sliced up by concentric, circular blades, each having its center on the Y-axis. Imagine that the blades are very close together. These blades will cut the "bowl" into very thin pieces as shown on the left below. We will approximate each piece by a "shell". A shell looks like an empty metal can with the top and bottom removed. An example of one such shell is shown on the right below.



Bowl-shaped figure sliced into concentric shells

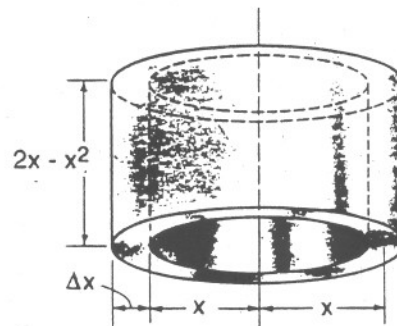


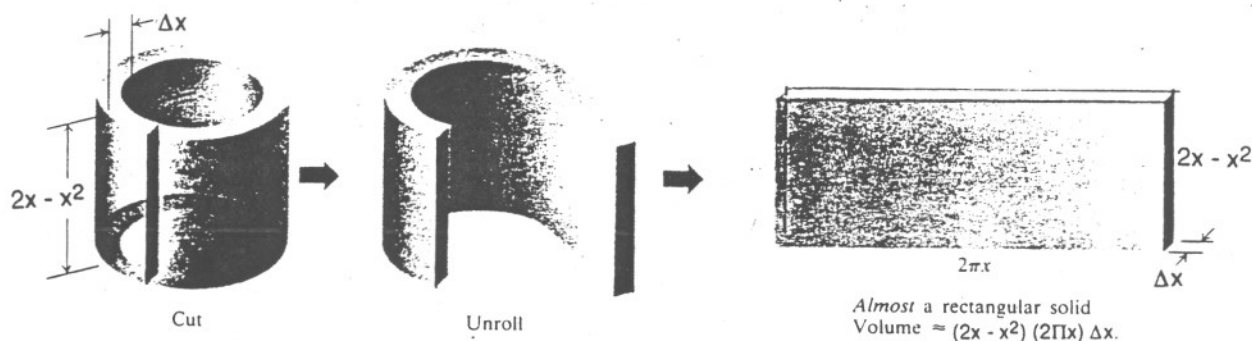
Illustration of one shell

We want to get the volume of a typical shell. The height is the difference of the y values for the two curves. Since the y value of the upper curve is $2x$ and the y value of the lower curve is x^2 , it follows that the height of the shell is $2x - x^2$.

The distance between the cutting blades could be measured along the X -axis, and so the thickness of the shell is Δx .

The radius of the shell is x , and so its circumference is $2\pi x$.

We will use a trick to approximate the volume of a shell. Imagine that the shell is cut and then flattened out to form a rectangular solid. (See diagram below.)



We know that the volume of a rectangular solid is the product of its height, length, and thickness. The height of the rectangle is $2x - x^2$ and the length of the rectangle is the circumference of the shell, which is $2\pi x$. The thickness is Δx . We can therefore conclude that the volume of this shell is $(2x - x^2)(2\pi x)\Delta x$. This simplifies to $2\pi(2x^2 - x^3)\Delta x$.

Volume of the Entire Solid

We must now add up the volumes of all the shells. We divided the interval from $x = 0$ to $x = 2$ into segments, each of length Δx . By adding up the volumes of all the concentric shells from $x = 0$ to $x = 2$ and letting Δx approach 0, we can express the volume of the entire bowl-shaped solid by the following integral.

$$2\pi \int_0^2 (2x^2 - x^3) dx.$$

By evaluating this integral, we can conclude that the volume of the solid is $\frac{8\pi}{3}$.