Math 207 Quiz #8 Nar

Directions: Please show all your work to receive credit. Relax. Good luck.

1. (6 points) Find all equilibrium solutions of each of the following systems. Classify each equilibrium solution as stable, unstable or neutrally stable. Also classify each as a node, spiral, or center.

a.

$$\frac{dx}{dt} = 2x - 3y$$

$$\frac{dy}{dt} = 3x + y$$

b.

$$\frac{dx}{dt} = 2x + 3y$$

$$\frac{dy}{dt} = 3x + 7y$$

2. (12 points) A building consists of two zones, A and B. Zone A is heated by a furnace, which generates 80,000 Btu/hr. The heat capacity of A is 1/4°C per thousand Btu. The time constant for heat transfer between each zone and the outside is 1 hour. The time constant between the two zones is 1/2 hour. The outside temperature is a constant 0°C. If the initial temperature of both zones is 20°C, express the temperature of each zone as a function of time. What happens to the temperature of each zone in the long run? Clearly define all variables with units.

3. An arms race between Mars and Jupiter is modeled by the system

$$\frac{dx}{dt} = -x + y - 2$$

$$\frac{dy}{dt} = 4x + 2y - 10$$

where x and y are the annual expenditures (in zillions of dollars) of Mars and Jupiter, respectively, and t is the number of years since 2156. Consider a negative expenditure to be a dismantling of weapons.

a. (4 points) Show that most of the time a runaway arms race occurs and determine the long-term ratio of annual expenditures between the two planets in this scenario.

b. (3 points) Find a possible set of initial expenditures (in the year 2156) for the countries so that a stabilized arms race occurs (x and y approach a constant as $t \to \infty$, but x and y themselves are not constant). Find the long-term annual expenditures for each planet in this scenario.

1. (a) $\frac{dx}{dt} = 2x - 3y$

$\frac{dy}{dt} = 3x + y$

Set:

$$\begin{vmatrix} 2-\lambda & -3 \\ 3 & 1-\lambda \end{vmatrix} = 0$$

$$2 - 3\lambda + \lambda^2 + 9 = 0$$

$$\lambda^2 - 3\lambda + 11 = 0$$

$$\lambda = \frac{3 \pm \sqrt{9-44}}{2} = \frac{3}{2} \pm \frac{\sqrt{35}}{2} i$$

$$x = C_1 e^{\frac{3}{2}} \cos\frac{\sqrt{35}}{2}t + C_2 e^{\frac{3}{2}} \sin\frac{\sqrt{35}}{2}t$$

$$\frac{dx}{dt} = \frac{3}{2}C_1 e^{\frac{3}{2}} \cos\frac{\sqrt{35}}{2}t - \frac{\sqrt{35}}{2}C_1 e^{\frac{3}{2}} \sin\frac{\sqrt{35}}{2}t + \frac{3}{2}C_2 e^{\frac{3}{2}} \sin\frac{\sqrt{35}}{2}t + \frac{\sqrt{35}}{2}C_2 e^{\frac{3}{2}} \cos\frac{\sqrt{35}}{2}t$$

$$y = \frac{1}{3}\left(2C_1 e^{\frac{3}{2}} \cos\frac{\sqrt{35}}{2}t + 2C_2 e^{\frac{3}{2}} \sin\frac{\sqrt{35}}{2}t - \frac{dx}{dt}\right)$$

Soln. $\begin{cases} x = C_1 e^{\frac{3}{2}} \cos\frac{\sqrt{35}}{2}t + C_2 e^{\frac{3}{2}} \sin\frac{\sqrt{35}}{2}t \\ y = \left(\frac{1}{2}C_1 - \frac{\sqrt{35}}{2}C_2\right) e^{\frac{3}{2}} \cos\frac{\sqrt{35}}{2}t + \left(\frac{1}{2}C_2 + \frac{\sqrt{35}}{2}C_1\right) e^{\frac{3}{2}} \sin\frac{\sqrt{35}}{2}t \end{cases}$

<u>Unstable spiral.</u>

1. (b) $\frac{dx}{dt} = 2x + 3y$

$\frac{dy}{dt} = 3x + 7y$

Set $\begin{vmatrix} 2-\lambda & 3 \\ 3 & 7-\lambda \end{vmatrix} = 0$ $\quad \begin{cases} (2-\lambda)V_1 + 3V_2 = 0 \\ 3V_1 + (7-\lambda)V_2 = 0 \end{cases}$

$14 - 9\lambda + \lambda^2 - 9 = 0$

$\lambda^2 - 9\lambda + 5 = 0$

$\lambda = \frac{9 \pm \sqrt{81-20}}{2} = \frac{9 \pm \sqrt{61}}{2}$

if $\lambda = \frac{9+\sqrt{61}}{2}$

$3V_1 + \left(\frac{14-9-\sqrt{61}}{2}\right)V_2 = 0$

$V_1 = \frac{\sqrt{61}-5}{6} V_2$

not necessary

if $\lambda = \frac{9-\sqrt{61}}{2}$

$3V_1 + \left(\frac{14-9+\sqrt{61}}{2}\right)V_2 = 0$

$V_1 = \frac{-5-\sqrt{61}}{6} V_2$

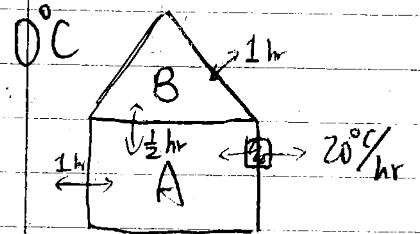
Soln: $\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{\frac{9+\sqrt{61}}{2}} \begin{bmatrix} 1 \\ \frac{\sqrt{61}-5}{6} \end{bmatrix} + c_2 e^{\frac{9-\sqrt{61}}{2}} \begin{bmatrix} 1 \\ \frac{-5-\sqrt{61}}{6} \end{bmatrix}$

Unstable node.

$$(80000\ \tfrac{\text{Btu}}{\text{hr}})(\tfrac{1}{4}\ {}^{\circ}\text{C}/1000\ \text{Btu}) = 20\ {}^{\circ}\text{C}/\text{hr}$$

(2) Let X be the temperature of zone A at time t hour

Y " " " " " B at time t hour



$X(0) = 20^{\circ}C$

$Y(0) = 20^{\circ}C$

$$\Rightarrow \begin{cases} \frac{dX}{dt} = 1(0-X) - 2(X-Y) + 20 \\ \frac{dy}{dt} = 1(0-Y) + 2(X-Y) \end{cases}$$

$$\Rightarrow \begin{cases} \frac{dX}{dt} = -3X + 2y + 20 \\ \frac{dy}{dt} = 2X - 3y \end{cases}$$

Solve

$$\Rightarrow \begin{cases} (D+3)X - 2Y = 20 \\ -2X + (D+3)Y = 0 \end{cases} \Rightarrow \begin{cases} -2(D+3)X + 4Y = -40 \\ -2(D+3)X + (D+3)^2 Y = 0 \end{cases}$$

$$\Rightarrow ((D+3)^2 - 4)Y = 40 \Rightarrow (D+5)(D+1)Y = 40$$

$$(D^2 + 6D + 5)Y = 40$$

$$\underline{Y = 8 + C_1 e^{-t} + C_2 e^{-5t}} \nearrow \frac{dy}{dt} = -C_1 e^{-t} - 5C_2 e^{-5t}$$

Then

$$X = \frac{1}{2}(-C_1 e^{-t} - 5C_2 e^{-5t} + 24 + 3C_1 e^{-t} + 3C_2 e^{-5t})$$

$$\Rightarrow \begin{cases} X = 12 + C_1 e^{-t} + C_2 e^{-5t} & X(0) = 20 \\ Y = 8 + C_1 e^{-t} + C_2 e^{-5t} & Y(0) = 20 \end{cases}$$

$$\Rightarrow \begin{cases} \overset{8}{\cancel{20}} = \cancel{12} + C_1 - C_2 \\ \overset{12}{\cancel{20}} = 8 + C_1 + C_2 \end{cases} \Rightarrow \begin{cases} C_1 - C_2 = 8 \\ C_1 + C_2 = 12 \end{cases} \Rightarrow \begin{cases} C_1 = 10 \\ C_2 = 2 \end{cases}$$

$$\Rightarrow \begin{cases} X = 12 + 10e^{-t} - 2e^{-5t} \\ Y = 8 + 10e^{-t} \underset{+}{-} 2e^{-5t} \end{cases}$$

As $t \to \infty$ $\Rightarrow \begin{cases} X \to 12^\circ C \\ Y \to 8^\circ C \end{cases}$

③ $\begin{cases} \frac{dx}{dt} = -x+y-2 \\ \frac{dy}{dt} = 4x+2y-10 \end{cases}$

(a)

$\Rightarrow \begin{cases} (D+1)x - y = -2 \\ -4x+(D-2)y = -10 \end{cases} \Rightarrow \begin{cases} 4(D+1)x - 4y = -8 \\ -4(D+1)x + (D-2)(D+1)y = (D+1)(-10) \end{cases}$

$\Rightarrow (D-2)(D+1)y - 4y = -18 \Rightarrow (D^2-D-6)y = -18$

So $y = 3 + C_1 e^{3t} + C_2 e^{-2t}$

$\Rightarrow \frac{dy}{dt} = 3C_1 e^{3t} - 2C_2 e^{-2t}$

$x = \frac{1}{4}(3C_1 e^{3t} - 2C_2 e^{-2t} - 6 - 2C_1 e^{3t} - 2C_2 e^{-2t} + 10)$

$= 1 + \frac{1}{4} C_1 e^{3t} - C_2 e^{-2t}$ ✓

$\Rightarrow \begin{cases} x = 1 + \frac{1}{4} C_1 e^{3t} - C_2 e^{-2t} \\ y = 3 + C_1 e^{3t} + C_2 e^{-2t} \end{cases}$

Since this solution has the saddle point, most of the time a runaway arms race occurs.

Long term ratio $\frac{x}{y} \sim \frac{1}{4}$

(3) (b) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_1 \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$

If $c_1 = 0$, then $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ as $t \to \infty$.

Choosing $c_1 = 0$ and $c_2 = 1$ gives

$x(0) = 0$

and $y(0) = 4$.

So if the initial expenditures for Mars and Jupiter are \$0 and \$4 zillion, respectively, a stabilized arms race occurs with

$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ as $t \to \infty$.